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Periodic Forests of Stunted Trees

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PERIODIC FORESTS OF STUNTED TREES

By J. C. P. MILLER

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[Plate 1]

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We may define a *forest of stunted trees* as follows:

Consider an infinite background of *nodes* at the vertices of an infinite plane tessellation of equilateral triangles, and start from a straight line of nodes at unit distance apart, which we shall consider as the *ground*; other parallel lines of nodes are then spaced at successive levels of linearly increasing heights above the ground. Any node may be *live* (if a tree passes through it) or *vacant* otherwise. Any live node may give rise to a *branch* to one or other or both of the two nearest nodes at the next higher level, but this growth is *stunted*, on either side, if the neighbouring node on that side is also live and could provide a branch to the same higher level node (this other branch is also stunted). Many of the figures in the paper show the type of forest that results.

The Introduction, §1, describes the origin of this idea, and §2 gives definitions and points out certain basic properties and ideas for combining forests and for separating them into simpler units. A variety of periodicities is discussed. In §3 a mathematical theory is developed in terms of generating functions expressed as power series. Sequences and forests are represented by ratios $\phi(t)/f(t)$ of polynomials with coefficients in $GF(2)$. A matrix formulation is also defined. The theory is developed in §4, so that periods and forests can be developed from those for basic sets having *irreducible* polynomials $f(t)$ as denominators, with co-prime numerators of lower degree. In §5, the determination of base- and row-periods for particular irreducible polynomials $f(t)$ is investigated as a preliminary to the enumeration of forests with given base-period n in §6, and of reflexive forests in §7. Further interesting properties, problems and applications are discussed in §8; it is intended to develop some of these in another paper.

The tables give enumerations and properties connected with sequences and forests generated by various polynomials $f(t)$ of low degree, culminating in table 5, which gives the numbers of forests with base periods up to 50, and table 6, which lists all individual forests with n up to 15.

Many of these forests are given in the diagrams, intended to bring out various symmetry properties and possible variations.

1. Introduction

Sierpinski, in his fascinating little book, *A selection of problems in the theory of numbers*, mentions the conjecture of Gilbreath (Sierpinski 1964, p. 35; Gilbreath 1958) which provided the inspiration for this particular field of research.

This conjecture concerns the successive absolute differences derived from the list of prime numbers, i.e. the table of $\Delta^r p_n$, in which we define

$$\Delta u_s = |u_{s+1} - u_s|.$$

The beginning of this table is given in table 1 (in which a zero is indicated by a dot). Gilbreath's conjecture is that the leading difference in each column is always unity. This has been verified for r up to 63418 (Killgrove & Ralston 1959). It seems one of those problems in number theory that are easy to state, but excessively difficult to prove.

Table 1. Absolute differences of primes

p_n																																	
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	2	·	2	2	2	2	2	2	·	·	·	·	·	·	2	2	·	2	2	·	·	2	2	2	·	·	·	2	2	·	2	·	
5	2	2	·	·	·	·	·	2	·	·	·	·	·	2	·	2	2	·	2	·	2	·	·	2	·	·	2	·	2	2	2		
7	4	2	·	·	·	·	2	2	·	·	·	·	2	2	2	·	2	2	2	2	2	·	2	2	·	2	2	2	·	·			
11	2	2	·	·	·	2	·	2	·	·	·	2	·	·	2	2	·	·	·	·	2	2	·	2	2	·	·	2	·				
13	4	2	·	·	2	2	2	2	·	·	2	2	·	2	·	2	·	·	·	2	·	2	2	·	2	·	2	2					
17	2	2	·	2	·	·	·	2	·	2	·	2	2	2	2	2	·	·	2	2	2	·	2	2	2	2	·						
19	4	2	2	2	·	·	2	2	2	2	2	·	·	·	·	2	·	2	·	·	2	2	·	·	·	2							
23	6	4	·	2	·	2	·	·	·	·	2	·	·	·	2	2	2	2	·	2	·	2	·	·	2								
29	2	4	2	2	2	2	·	·	·	2	2	·	·	2	·	·	·	2	2	2	2	2	·	2									
31	6	2	·	·	·	2	·	·	2	·	2	·	2	2	·	·	2	·	·	·	·	2	2										
37	4	2	·	·	2	2	·	2	2	2	2	2	·	2	·	2	2	·	·	·	2	·											
41	2	2	·	2	·	2	2	·	·	·	·	2	2	2	2	·	2	·	·	2	2												
43	4	2	2	2	2	·	2	·	·	·	2	·	·	·	2	2	2	·	2	·													
47	6	·	4	4	2	2	2	·	·	2	2	·	·	2	·	·	2	2	2														
53	6	4	·	2	·	·	2	·	2	·	2	·	2	2	·	2	·	·															
59	2	4	2	2	·	2	2	2	2	2	2	2	·	2	2	2	·																
61	6	2	·	2	2	·	·	·	·	·	·	2	2	·	·	2																	
67	4	2	2	·	2	·	·	·	·	·	2	·	2	·	2																		
71	2	4	2	2	2	·	·	·	·	2	2	2	2	2																			
73	6	2	·	·	2	·	·	2	2	·	·	·	·																				
79	4	2	·	2	2	·	·	2	2	·	·	·																					
83	6	2	2	·	2	·	2	4	2	·	·																						
89	8	4	2	2	2	2	6	6	2	·																							
97	4	2	·	·	·	8	·	8	2																								
101	2	2	·	·	8	8	8	6																									
103	4	2	·	8	·	·	2																										
107	2	2	8	8	·	2																											
109	4	10	·	8	2																												
113	14	10	8	6																													
127	4	2	2																														
131	6	4																															
137	2																																
139																																	

In an attempt to simplify the problem into something capable of reasonable mathematical treatment, attention is immediately directed to the fact that most of the table (apparently all of it as r gets large) is composed of 2's and 0's with the exception of the leading row of ones.

A first variation is to consider a different table, in which the first column still depends on the primes, but consists of units (in place of the 2's) and zeros only. The first choice was to take

TABLE 2. ABSOLUTE DIFFERENCES OF b_n

$b_n = 1$, $2n+1$ prime; $b_n = 0$, $2n+1$ composite.

2n+1	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79
b_n	1	1	1	1	·	1	1	·	1	1	·	1	·	·	1	1	·	·	1	·	1	1	·	1	·	·	1	·	·	1	1	·	·	1	·	1	1	·	·	1
	·	·	·	1	1	·	1	1	·	1	1	1	·	1	·	1	·	1	1	1	·	1	1	1	·	1	1	·	1	·	1	·	1	1	1	·	1	·	1	
	·	·	1	·	1	1	·	1	1	·	·	1	1	1	1	1	1	·	·	1	1	·	·	1	1	·	1	1	1	1	1	1	·	·	1	1	1	1		
	·	1	1	1	·	1	1	·	1	·	1	·	·	·	·	·	1	·	1	·	1	·	1	·	1	1	·	·	·	·	·	1	·	1	·	·	·			
	1	·	·	1	1	·	1	1	1	1	1	·	·	·	·	1	1	1	1	1	1	1	1	1	·	1	·	·	·	·	1	1	1	1	·	·				
	1	·	1	·	1	1	·	·	·	·	1	·	·	·	1	·	·	·	·	·	·	·	·	1	1	1	·	·	·	1	·	·	·	1	·					
	1	1	1	1	·	1	·	·	·	1	1	·	·	1	1	·	·	·	·	·	·	·	1	·	·	1	·	·	1	1	·	·	1	1						
	·	·	·	1	1	1	·	·	1	·	1	·	1	·	1	·	·	·	·	·	·	1	1	·	1	1	·	1	·	1	·	1	·							
	·	·	1	·	·	1	·	1	1	1	1	1	1	1	1	·	·	·	·	·	1	·	1	1	·	1	1	1	1	1	1	1								
	·	1	1	·	1	1	1	·	·	·	·	·	·	·	1	·	·	·	·	1	1	1	·	1	1	·	·	·	·	·	·									
	1	·	1	1	·	·	1	·	·	·	·	·	·	1	1	·	·	·	1	·	·	1	1	·	1	·	·	·	·	·										
	1	1	·	1	·	1	1	·	·	·	·	·	1	·	1	·	·	1	1	·	1	·	1	1	1	·	·	·	·											
	·	1	1	1	1	·	1	·	·	·	·	1	1	1	1	·	1	·	1	1	1	1	·	·	1	·	·	·												
	1	·	·	·	1	1	1	·	·	·	1	·	·	·	1	1	1	1	·	·	·	1	·	1	1	·	·													
	1	·	·	1	·	·	1	·	·	1	1	·	·	1	·	·	·	1	·	·	1	1	1	·	1	·														
	1	·	1	1	·	1	1	·	1	·	1	·	1	1	·	·	1	1	·	1	·	·	1	1	1															
	1	1	·	1	1	·	1	1	1	1	1	1	·	1	·	1	·	1	1	1	·	1	·	·																
	·	1	1	·	1	1	·	·	·	·	·	1	1	1	1	1	1	·	·	1	1	1	·																	
	1	·	1	1	·	1	·	·	·	·	1	·	·	·	·	·	1	·	1	·	·	1																		
	1	1	·	1	1	1	·	·	·	1	1	·	·	·	·	1	1	1	1	·	1																			
	·	1	1	·	·	1	·	·	1	·	1	·	·	·	1	·	·	·	1	1																				
	1	·	1	·	1	1	·	1	1	1	1	·	·	1	1	·	·	1	·																					
	1	1	1	1	·	1	1	·	·	·	1	·	1	·	1	·	1	1																						
	·	·	·	1	1	·	1	·	·	1	1	1	1	1	1	1	·																							
	·	·	1	·	1	1	1	·	1	·	·	·	·	·	·	1																								
	·	1	1	1	·	·	1	1	1	·	·	·	·	·	1																									
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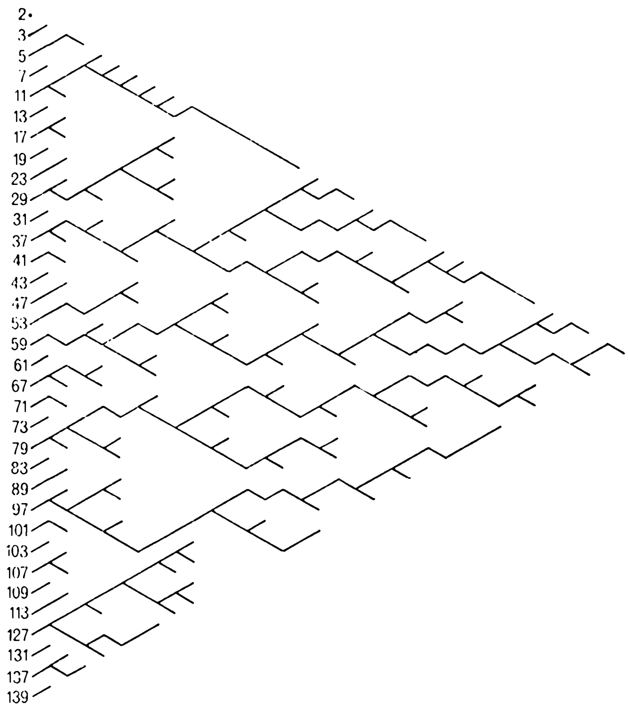

FIGURE 1. Trees from absolute differences of primes.

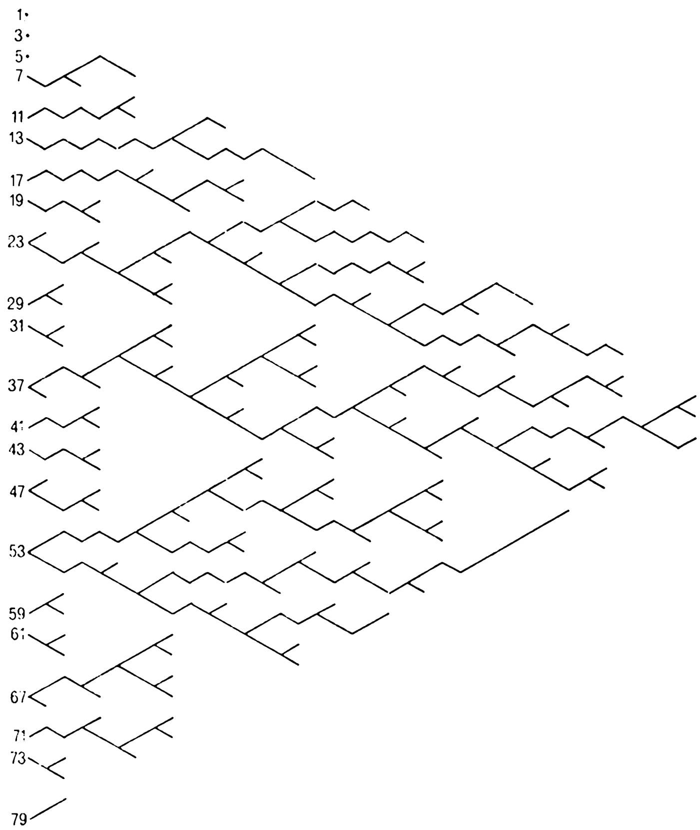

FIGURE 2. Trees from prime roots.

$u_n = 0$ for n composite, $u_n = 1$ for n prime, and to construct this table. It has, however, $u_n = 0$ for all even n, and, as we shall see later (§4.1), there is no significant change in structure if we confine the table to odd n.

Table 2 gives the values of $\Delta^r b_n$, where $b_n = 1$ if $2n+1$ is prime, $b_n = 0$ if $2n+1$ is composite. Again zero is replaced by a dot.

A branching structure is clear in table 2, and this suggests trees. Either table is immediately converted into tree form by replacing each non-zero difference by a live node, and joining it to the *larger* of the two quantities from which it was derived.

Figures 1 and 2 give illustrations of the trees corresponding to tables 1 and 2. In the figures each 'live' (non-zero) node has a line to or through it, each 'vacant' (zero) node is blank, and the tree 'grows' to the right.

These two illustrations exhibit many of the characteristics of non-periodic forests of this type, and give a basis for definitions of these characteristics. The diagrams clearly have the form of mathematical trees, with nodes arranged in lines (one at each order of differences $\Delta^r u_n$) and with 0, 1 or 2 branches extending 'upwards' (i.e. to the right in figures 1 and 2). At each node reached by a branch, two further branches may arise, but either branch is 'stunted' if there is a live node as an immediate neighbour at the same level on the corresponding side (i.e. the corresponding branches on *both* the neighbouring nodes fail to grow, though any branch directed away from the neighbouring node is unaffected by the particular juxtaposition considered here).

In figure 1 the effect of early differences exceeding 2 is apparent in the special form of the trees in the first few nodes near the root. This effect will not be studied here, though Gilbreath's conjecture depends, in fact, on the non-persistence of differences 4 or more right up to the leading edge.

Both figures exhibit the interesting characteristic phenomenon of triangular *clearings* (it seems reasonable that large clearings may encourage the persistence of large differences).

In these irregular forests, many of the trees come to a complete end of growth, others extend eventually to the leading edge.

The diagrams may also be pictured in terms of *rivers*, flowing from right to left. The clearings then become *deserts* (it is perhaps a little odd that springs at the edges of deserts arise in straight lines at equally spaced points!). With this picture one may also refer to *watersheds* corresponding to lines of separation or stunting between neighbouring trees (*stunt lines*).

In figure 2 the pattern is still irregular, but suggests immediately the consideration of *periodic* patterns, in the hope that these may turn out to be more amenable to mathematical discussion. This turns out to be the case; the study of sequences produced by binary feedback shift registers, with a theory involving operations in the Galois field GF(2) provides most of the answers.

2. General ideas

2.1. *Definitions and elementary properties*

We consider *forests* of *trees* with *roots* that are allowed to occur only at equally spaced points along a particular straight line, which we shall take as one coordinate axis (x). Figure 3 shows part of such a forest. The background of possible *nodes* consists of the vertices of a network of equilateral triangles, which includes the points $(r, 0)$, r an integer, along the x axis; these are the possible roots. Other nodes are given by $(r+\frac{1}{2}s, \frac{1}{2}\sqrt{3}\, s)$ with r, s integers; it is sometimes convenient to confine r, s to non-negative integers, but they can also be imagined to take all integer values, corresponding to a forest of 'trees' without roots, but extending indefinitely 'downwards', i.e. towards $y \to -\infty$. We shall use $[r, s]$ as a short notation for $(r+\frac{1}{2}s, \frac{1}{2}\sqrt{3}s)$, and note that for $r, s > 0$ we are confined to the polar sector $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

A node $[r, s]$ is *live* if it is connected by a *branch* to one or other of the nodes $[r, s-1]$ or $[r+1, s-1]$. This applies except if $s = 0$, i.e. except to a root; a *root* is *live* if it is permitted by the initial conditions to give rise to a tree (though growth from one of these roots may nevertheless be completely stunted as described below). All other nodes will be called *vacant*.

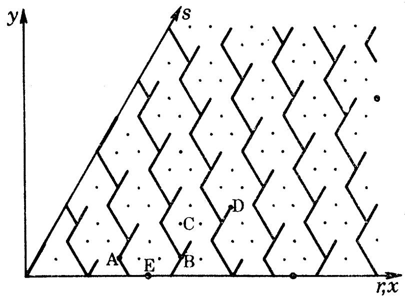

FIGURE 3

A live node $[r, s]$ may give rise to one or two *branches*, connecting to one or other or both of the nodes $[r, s+1]$, $[r-1, s+1]$, that is to branches along the sides of the equilateral triangle with $[r, s]$ as vertex and base on the side *away* from the x axis. The growth is, however, *stunted*, and not permitted to $[r, s+1]$ if $[r+1, s]$ is also live, and (independently) not permitted to $[r-1, s+1]$ if $[r-1, s]$ is also live. Thus at A $[4, 1]$ in figure 3, $[3, 1]$ is live and $[5, 1]$ is vacant, so only the branch $[4, 1]$ to $[4, 2]$ can grow. At B $[7, 1]$ both branches can grow. At C, which is vacant, and at D, which has both neighbours live, no growth is allowed.

In general, if the sign of r is unrestricted, trees may be of indefinite height, and will be if the forest is periodic; if r is restricted to be positive, trees will all, in general, eventually reach the edge $r = 0$ and terminate. We say 'in general' because if s is restricted to be $\geqslant 0$, the point E, for example, is regarded as 'live' to give the same period (7 units in figure 3) and shape in all rows, but is detached from its appropriate tree, which it would normally meet on $s = -1$. Such accidental detachment can easily extend to higher levels of s with greater periods than 7, through all originate at $s = 0$.

We shall confine attention to cases where live roots of trees (which we 'tag' with a 1) occur in a periodic arrangement along the x axis, vacant root positions will be tagged with 0. This leads to a binary succession of root-tags characteristic of each forest. For example figure 2 yields the (non-periodic) sequence

11110110110100110010110100100 11 etc.

which is sometimes more easily assimilated with . in place of 0, thus

1111.11.11.1..11..1.11.1..1..11 etc.

Clearings have been mentioned in the introduction. They play a useful part in the theory, particularly in organizing the actual identification and construction of distinct forests (see Miller 1968). The *size* of a clearing is measured by the longest row of vacant nodes within it. Thus the largest clearing in figure 2 had size 8. A triangle of three adjacent live nodes (at two levels) is regarded as a 'clearing' of size 0; a clearing of size 1 is a regular hexagon, clearings of larger sizes are hexagons with three edges of unit length, alternating with three longer edges (we imagine here the polygon to be completed by joining adjacent nodes all round the clearing, including, for example, a join between $[r-1, s]$ and $[r, s]$, if both live, parallel to the x axis; figure 4 shows the forest of figure 3 thus joined to form a *tessellation*).

2.2. *Triangular symmetry of aspect*

It will be observed that all clearings of zero size are triangles with a vertex towards the x axis, and consisting of three live nodes. Other triangles, still with vertex towards the x axis, may have 0, 1, or 2 live nodes; triangles of this aspect, with vertex towards the x axis when $s > 0$, have been mentioned above as indicating possible directions of growth from a live node—we shall call them *branch-* or *growth-triangles*, or B-*triangles*.

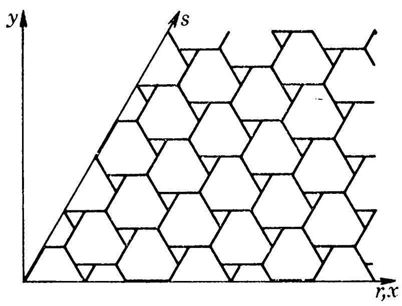

FIGURE 4

On the other hand, unit triangles of the opposite aspect, i.e. with vertex *away* from the axis when $s > 0$ must always have 0 or 2 live nodes, by the rules of restriction of growth by stunting. We shall thus refer to these later as *check-* or *stunt-triangles*, or C-*triangles*. The existence of an even number of live nodes is a symmetry property of a C-triangle, that is, it is independent of the side taken as base. Thus we could take the line $r = 0$ to give a system of roots (corresponding to the live nodes) and apply the stunting rules to growth to the right in a direction perpendicular to $r = 0$, exactly the same complete system of live nodes would result in the sector $0 \leqslant \theta \leqslant \frac{1}{3}\pi$. We could likewise take a line of roots along $r+s = R$, a constant, and have stunted growth towards the origin—again the same system of live nodes would result in overlapping regions. Figure 8 (p. 78) illustrates this—the triangular symmetry is apparent (though of course the actual lines used as branches differ in the three forests, since a branch cannot be parallel to the root-line). As we shall see later, the three forests may be completely distinct with differing periods.

We note here that it is useful to have a name, *copse*, for trees enclosed in a finite triangle $r = 0$, $s = 0$, $r+s = R$.

2.3. *Addition of forests*

A consequence of paramount importance based on the fact that C-triangles always have an even number of live nodes, and that this property alone characterizes a complete forest, is that any two forests may be combined or 'added modulo 2'. The forests are placed node to node, and the 'sum' of two live nodes, or of two vacant nodes, is a vacant node; the sum of two unlike nodes is a live node. The number of live nodes in every C-triangle remains even, and a proper periodic forest of stunted trees results. If, as suggested above, we tag live nodes with 1 and vacant nodes with 0, the construction becomes *addition modulo* 2.

2.4. *Alternation*

Consider now the array of nodes tagged as in diagram 1 with each $w_{r,s}$ either 0 or 1. We see that, modulo 2,

$w_{0,3}$ $w_{1,3}$

$w_{0,2}$ $w_{1,2}$ $w_{2,2}$

$w_{0,1}$ $w_{1,1}$ $w_{2,1}$

$w_{0,0}$ $w_{1,0}$ $w_{2,0}$ $w_{3,0}$

Diagram 1

$$\begin{aligned} w_{0,2} &= w_{0,1}+w_{1,1} \\ &= w_{0,0}+w_{1,0}+w_{1,0}+w_{2,0} \\ &= w_{0,0}+w_{2,0} \end{aligned} \tag{2.4.1}$$

and generally, that
$$w_{r,s+2} = w_{r,s}+w_{r+2,s}. \tag{2.4.2}$$

It follows that the $w_{2r,2s}$ $(r, s = 0, 1, 2, \ldots)$ form a forest on their own, obtained by taking *alternate nodes suitably from alternate rows*. Likewise each of the sets $w_{2r,2s+1}$, $w_{2r+1,2s}$, $w_{2r+1,2s+1}$ similarly gives a forest.

We call this process *alternation* and, by it, each forest gives four separate subforests, each with the standard rule of formation.

The process can clearly be repeated, and also reversed. That is, we can build up a forest from four subforests, which are not, however, all arbitrary. In fact, two may be chosen arbitrarily—the other two are then determinate. This may be seen by noting that choice of two forests gives a complete row of nodes in one of the three possible orientations of the background, and from this the rest can be reconstructed.

2.5. *Forest layers*

We can also relate forests in parallel layers, as follows:

Each C-triangle has an even number of live nodes, but each B-triangle may have any number, 0, 1, 2, or 3. Erect on each B-triangle a regular tetrahedron, above the plane of the original forest, and label the new vertex with the sum, mod 2, of the tags on the three vertices of the B-triangle that forms its base. Then the array of new vertices of all the tetrahedra so found gives a background of live nodes that forms a forest.

$w_{0,3}$ $w_{1,3}$ $w_{2,3}$

$W_{0,2}$ $W_{1,2}$

$w_{0,2}$ $w_{1,2}$ $w_{2,2}$

$W_{0,1}$ $W_{1,1}$ $W_{2,1}$

$w_{0,1}$ $w_{1,1}$ $w_{2,1}$ $w_{3,1}$

$W_{0,0}$ $W_{1,0}$ $W_{2,0}$

$w_{0,0}$ $w_{1,0}$ $w_{2,0}$ $w_{3,0}$

Diagram 2

To see this, consider diagram 2, in which $w_{r,s}$ refers to the initial layer, and $W_{r,s}$ to the new layer. We have
$$W_{r,s} = w_{r+1,s}+w_{r,s+1}+w_{r+1,s+1} \tag{2.5.1}$$
and from the diagram
$$\begin{aligned} W_{0,0}+W_{1,0}+W_{0,1} = &\left.\begin{aligned} &w_{1,0}+w_{0,1}+w_{1,1} \\ +&w_{2,0}+w_{1,1}+w_{2,1} \\ +&w_{1,1}+w_{0,2}+w_{1,2} \end{aligned}\right\} \text{adding horizontally (mod 2)} \\ = &\ 0 \quad \text{adding vertically (mod 2).} \end{aligned} \tag{2.5.2}$$

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Generally $$W_{r,s}+W_{r+1\ s}+W_{r,s+1} = 0 \pmod 2. \tag{2.5.3}$$

We note also that
$$\begin{aligned} W_{0,0} &= w_{0,1}+w_{1,0}+w_{1,1} \\ &= w_{0,0}+w_{1,0}+w_{1,0}+w_{1,0}+w_{2,0} \\ &= w_{0,0}+w_{1,0}+w_{2,0} \end{aligned} \tag{2.5.4}$$

so that generally $$W_{r,s} = w_{r,s}+w_{r+1,s}+w_{r+2,s} \pmod 2. \tag{2.5.5}$$

2.6. *Periodicity*

There are several periodicities connected with forests and the corresponding arrays of live nodes. We shall now suppose the array extended indefinitely in all directions, and shall single out the lines $s = 0$, $r = 0$, $r+s = 0$ as the 'root-lines' of the three forests connected with a particular array of live nodes.

First there is the (least) *ground-* or *base-period* (B-*period*), n_1, of the original set of roots assumed along $s = 0$. All subsequent lines of constant s clearly have n_1 as a period, and the number of possible variations (at most 2^{n_1}) is finite; repetition must therefore occur, and the succession of rows becomes ultimately periodic. There will thus be periods along both $r = 0$ and $r+s = 0$, these need not be equal to n_1, nor to one another. We thus have three (least) B-*periods* n_1, n_2, n_3. We shall also refer to a *general base-period*, which must be a multiple of the corresponding B-period, though not necessarily equal to it. The B-periods n_1, n_2, n_3 have a least common multiple T. Clearly $[r, s]$, $[r+T, s]$, $[r, s+T]$ lead to identical tree structure thereafter (i.e. for increasing s), for r, s sufficiently large for periodicity to have become established. We thus have an overall *triangular period* or T-*period*; this period is both a base-period and a row-period and the complete pattern is one in which the rhombus, with these and $[r+T, s+T]$ as vertices, is repeated as a whole; ($[r+T, s-T]$ or $[r-T, s+T]$ would do equally well as fourth corner). This will be the least period involving a pair of large *equilateral* triangles, and exhibits the basic lattice structure. There may be smaller lattice cells consisting of pairs of isosceles triangles or of scalene triangles.

There may be a shorter row-period associated with the repetition of rows of nodes. Suppose that n_2 is the period along the axis of s, then the row $s = n_2$ repeats the row $s = 0$, in a purely periodic forest, with live nodes for exactly the same values of r. There may be, however, a row with $s = m_1 < n_2$, such that, for $s = m_1$, the state of the node at $r' = r+\rho_1$, ρ_1 constant, is exactly that for r when $s = 0$; that is the row at $s = 0$ is repeated for $s = m_1$, but translated by ρ_1 in r. In other words, the rows $s = 0$, $s = m_1$ exhibit the same *cycle*. This may happen in any or all of the three row-directions, corresponding to the three ground- or root-directions $s = 0$, $r = 0$, $r+s = 0$, and yielding three (least) *row-* or R-*periods*, m_1, m_2, m_3 respectively.

It is easily seen that the number of distinct unit-triangles (an equal number C for each kind, B-triangles and C-triangles), is given by
$$2C = 2n_1 m_1 = 2n_2 m_2 = 2n_3 m_3 \tag{2.6}$$
and that the unit basic lattice *cells* must have this area, which is thus a submultiple of $2T^2$.

Corresponding points in successive repetitions of a cycle in distinct rows may be translated by various amounts in the x direction; the precise amount is of some importance in the theory. If a cycle starts at $x = r$, the corresponding point after a row-period T starts at $x = r+\frac{1}{2}T$. There may, however, be an earlier repetition, after S rows for the earliest such repetition, which also has the corresponding cycle starting with $x = r+\frac{1}{2}T$. Since the B-period n_1 divides T, this implies that both repetitions, after S rows or after T rows, have the same cycle start also at $x = r+\frac{1}{2}n_1$. The

S-cycle thus has a period S that is an odd submultiple of T. (We shall see later that all periods associated with a forest background have the *same* power of 2 as a factor.)

The S-*period* or *symmetric period* is the same for any cycle and its reversed cycle, with the same corresponding starting point at $x = r + \frac{1}{2}n_1$ after S rows; it corresponds to a lattice cell consisting of a pair of isosceles triangles. The (least) R-*period*, m_1, may again be a proper submultiple of the S-period, but will then *not* have $x = r + \frac{1}{2}n_1$, but $x = r + \frac{1}{2}m_1 + \rho_1$, and the reversed cycle will also lead to an R-period, for the same r, with a different corresponding start at $x = r + n_1 - \frac{1}{2}m_1 - \rho_1$. This corresponds to a lattice-cell of two scalene triangles. For symmetric cycles, whence the name, the S-period is always its R-period as well. There may of course be different S-periods S_1, S_2, S_3 for the three possible root-lines.

When no confusion can occur, we shall use n, m, S in place of n_1, m_1, S_1.

We may also have *alternation periods*, and *layer periods*.

2.7. *Cyclic representation on cylinder or torus*

The periodicity can be viewed in another way. Consider a single strip of a forest, of width N in x, where N is a root-period. This can be wrapped round a cylinder of perimeter N units, and periodicity is then represented by progressing round the cylinder, corresponding to the complete range $-\infty < r < \infty$. Edge effects are thus eliminated. A useful period to take is $N = T$; the period in height along the cylinder means that the root-circle is repeated after T steps upwards, but turned through π, or half a perimeter.

We may also imagine this repeated ring identified with the original one, giving a representation on a torus (distorted in the normal three dimensions). Periods may then be made to correspond to circuits of the outer ring, to circuits of the centre hole, and to circuits of both simultaneously. Each of these circuits corresponds to a circle that can be inscribed on a torus, and which is not deformable into either of the others.

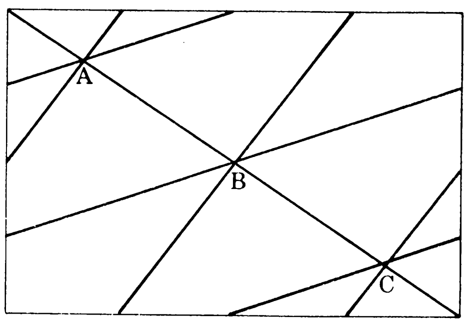

FIGURE 5

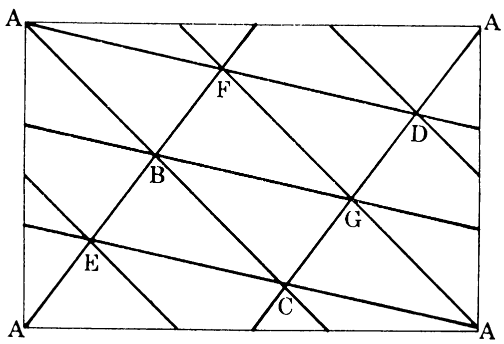

FIGURE 6

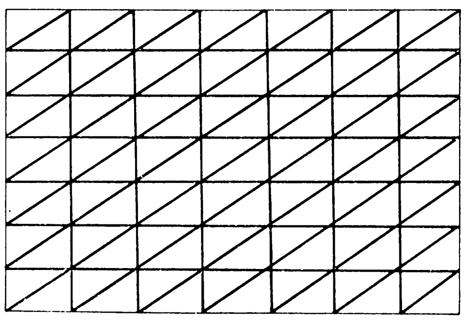

FIGURE 7

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The same process of representation on a torus can be applied to a single lattice cell of $2nm$ unit-triangles, though this may involve spiral lines of nodes, rather than the circles possible when a cell of $2T^2$ unit triangles is used. The visualization is easily obtained by distorting a suitable single cell, and we give in figures 5 to 7 the conventional rectangular diagrams for $n = 3$, $m = 1$, for $n = 7$, $m = 1$, and for $n = 7$, $m = 7$. Figure 6 corresponds to the forest of figure 3.

2.8. *Purpose of the paper*

The main purpose of this paper is to intiate a study of the possible periodic forests, and the triangular symmetry of aspect, and the possibility of adding forests, coupled with the general theory of periodic binary sequences (see Selmer 1966) which play a major part. There is also considerable interest in the corresponding plane-fillings by regular pattern or tessellations of special polygons as exhibited in figure 4.

3. Generating functions

The theory of periodic binary sequences has been well developed in connexion with the use of binary shift registers, using operations in binary non-carry arithmetic, i.e. in integers mod 2, or in the finite Galois field GF (2) (see, for instance, Selmer 1966, where the developments set out by various writers are exhibited and coordinated).

3.1. *Definitions*

We consider first the series of roots along the x axis. Starting from the origin, we define a generating function

$$G_0(t) = \sum_0^\infty w_{r,0} t^r \tag{3.1.1}$$

in which $w_{r,0} = 1$ for a live root, $w_{r,0} = 0$ for a vacant root position. For a sequence of period n, clearly

$$G_0(t) = \left(\sum_0^{n-1} w_{r,0} t^r\right)(1+t^n+t^{2n}+\dots)$$

$$= \frac{\sigma_0^*(t)}{1+t^n} \tag{3.1.2}$$

where
$$\sigma_0^*(t) = \sum_0^{n-1} w_{r,0} t^r$$

is constructed from the first period, and

$$\sum_{j=0}^\infty t^{jn} = (1-t^n)^{-1} = (1+t^n)^{-1}$$

with coefficients in GF (2).

If we now express the fraction in its lowest terms, with coefficients in GF (2), we find

$$G_0(t) = \frac{\phi_0^*(t)}{f^*(t)} \tag{3.1.3}$$

where
$$F^*(t) = (\sigma_0^*(t),\ 1+t^n) \tag{3.1.4}$$

has been cancelled. In the terminology of Selmer (1966) $f(t)$ is the *minimum polynomial* of the sequence $(w_{r,0})$.

3.2. *Digression on notation*

We use in the main the notation of Selmer (1966), in which $f^*(t) = t^j f(1/t)$, where $\deg(f) = j$; thus $f^*(t)$ and $f(t)$ are reciprocal polynomials. We adopt the following conventions in connexion with the representation of polynomials. Following Selmer we write

$$f(t) = t^k + c_{k-1}t^{k-1} + c_{k-2}t^{k-2} + \dots + c_1 t + c_0 \tag{3.2.1}$$

for a polynomial of degree k, in which $c_0 \neq 0$. We write the reciprocal polynomial

$$f^*(t) = 1 + c_{k-1}t + c_{k-2}t^2 + \ldots + c_1 t^{k-1} + c_0 t^k \tag{3.2.2}$$

with coefficients *in the same order*. We shall frequently use the detached coefficients as a 'number', which will be written as

$$1c_{k-1}c_{k-2}\ldots c_1 c_0 \tag{3.2.3}$$

whether for $f(t)$ or for $f^*(t)$; the polynomial intended will be indicated unless clear from the context.

Likewise for numerators $\phi_s^*(t)$ similar conventions are used. Zero coefficients are, however, now allowed in any position, including first and last, and the polynomial is of *formal degree* $k-1$.

$$\phi(t) = \phi_{k-1}t^{k-1} + \phi_{k-2}t^{k-2} + \ldots + \phi_1 t + \phi_0 \tag{3.2.4}$$

$$\phi^*(t) = \phi_{k-1} + \phi_{k-2}t + \ldots + \phi_1 t^{k-2} + \phi_0 t^{k-1}. \tag{3.2.5}$$

If the sequence of coefficients in $f(t)$ is reversed, so that the number in (3.2.3) becomes

$$c_0 c_1 \ldots c_{k-1} 1$$

(note that $c_0 = 1$ always in GF (2)) we get a distinct polynomial for purposes of discussion. This polynomial has, of course, many properties in common with those of the original polynomial, but it is useful to keep clear in this way the distinction between the use of a specific polynomial as a primary polynomial, and as the reciprocal of another.

Likewise, $\sigma^*(t) = s_0 + s_1 t + s_2 t^2 + \ldots + s_{n-1}t^{n-1}$ and $\sigma(t) = t^{n-1}\sigma^*(1/t)$ will also be written in the natural order $s_0 s_1 s_2 \ldots s_{n-1}$, as Selmer does, with suffixes in reverse order.

3.3. *Relation between generating functions for successive rows*

For each line of nodes with constant s there is a generating function

$$G_s(t) = \frac{\sigma_s^*(t)}{1+t^n} \tag{3.3.1}$$

similar to (3.1.2), and with the same n.

Consider now the relation between $G_s(t)$ and $G_{s+1}(t)$. Each live node $[r, s]$, $w_{r,s} = 1$, yields contributions to two successors $w_{r,s+1}$ and $w_{r-1,s+1}$; stunting corresponds precisely to addition mod 2 for the contributions from two predecessors. Thus, *except for the end-effect near* $r = 0$,

$$G_{s+1}(t) = (1+t^{-1})\,G_s(t). \tag{3.3.2}$$

The most convenient way to allow for the end-effect, which corresponds to a term t^{-1} at the beginning of a sequence of otherwise non-negative powers, is just to omit this term. The multiplication with the following omission is exactly equivalent to an operation on $(1+t)\,\sigma_s^*(t)$, mod $(1+t^n)$ to remove the *constant* term, in favour of one in t^n, followed by division by t. The net result is that $\sigma_s^*(t) = (1+t^{-1})^s\,\sigma_0^*(t)$ is reduced, mod $(1+t^n)$, to remove all negative powers and to finish with a polynomial of degree less than n.

We thus have, generally

$$\sigma_s^*(t) = (1+t^{-1})^s\,\sigma_0^*(t) \quad \bmod (1+t^n). \tag{3.3.3}$$

Also, using (3.1.4) and the fact that $t \nmid f^*(t)$ we have, *provided that* $(1+t) \nmid f^*(t)$,

$$\begin{aligned}(t^s\sigma_s^*(t),\ 1+t^n) &= ((1+t)^s\,\sigma_0^*(t),\ 1+t^n)\\ &= F^*(t)\,((1+t)^s\,\phi_0^*(t), f^*(t))\\ &= F^*(t).\end{aligned}$$

Hence
$$G_s(t) = \frac{\sigma_s^*(t)}{1+t^n} = \frac{\phi_s^*(t)}{f^*(t)} \tag{3.3.4}$$
in which
$$\sigma_s^*(t) = F^*(t)\,\phi_s^*(t) \quad \text{and} \quad (\phi_s^*(t), f^*(t)) = 1$$
and
$$\phi_s^*(t) = (1+t^{-1})^s\,\phi_0^*(t) \quad \bmod f^*(t). \tag{3.3.5}$$

We shall see in §4.3 that a factor $(1+t)^i$ in $f^*(t)$ causes the corresponding forest to have a non-periodic trunk of not more than i rows, before periodicity is established. We can then summarize the foregoing remarks to give:

In a purely periodic forest, a factor $(1+t)$ cannot occur in the denominator $f^*(t)$, which is the same for all rows, when $G_s(t)$ is expressed in lowest terms.

Alternatively:

A purely periodic forest consists entirely of sequences for which the same polynomial $f(t)$ is minimum polynomial. We shall call this polynomial the minimum generating polynomial for the forest.

3.4. *A matrix formulation*

No generating function in two, or perhaps three, variables has yet been constructed that treats the three forests of a background in a uniform fashion, and thereby exhibits the relationship between them.

An effective matrix formulation has, however, been suggested in discussion by F. L. Bauer. Consider a copse of size $n+1$, with base sequence $a_0, a_1, \ldots, a_n$, with sequence $b_0, b_1, \ldots, b_n$, where b_i is at the node $[r, s] = [n-i, i]$, and with $c_0, c_1, \ldots, c_n$ having c_i at node $[n-i, n-i]$, see diagram 3.

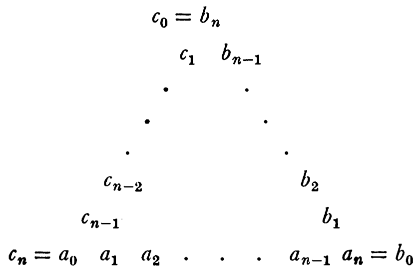

Diagram 3

Then, using the operator $Ea_i = a_{i+1}$, we have

$$\left.\begin{aligned} c_n &= a_0 \\ c_{n-1} &= a_0 + a_1 = (1+E)\,a_0 \\ c_{n-2} &= a_0 + a_2 = (1+E^2)\,a_0 = (1+E)^2 a_0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{n-i} &= (1+E)^i\,a_0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_0 &= (1+E)^n\,a_0. \end{aligned}\right\} \tag{3.4.1}$$

We now write $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ for the $(n+1)$-vectors

$$(a_0, a_1, \ldots, a_n)^T, \quad (b_0, b_1, \ldots, b_n)^T, \quad (c_0, c_1, \ldots, c_n)^T,$$

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and

$$A = A_{n+1} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & \\ 1 & 0 & 1 & & \\ 1 & 1 & & 0 & \\ 1 & & & & \end{pmatrix} \tag{3.4.2}$$

for the finite 'Pascal' matrix of binomial coefficients in which the jth element ($j = 0, 1, \ldots, n$) in the ith row ($i = 0, 1, \ldots, n$) is the coefficient of t^j in $(1+t)^{n-i}$, reduced mod 2. Then the equation (3.4.1) can be written as

$$\left.\begin{aligned} \boldsymbol{c} &= \boldsymbol{A}\boldsymbol{a} \\ \text{and similarly} \qquad \boldsymbol{b} &= \boldsymbol{A}\boldsymbol{c}, \quad \boldsymbol{a} = \boldsymbol{A}\boldsymbol{b} \end{aligned}\right\} \tag{3.4.3}$$

whence
$$\boldsymbol{c} = \boldsymbol{A}\boldsymbol{a} = \boldsymbol{A}^2\boldsymbol{b} = \boldsymbol{A}^3\boldsymbol{c}$$

and
$$\boldsymbol{A}^3 = \boldsymbol{I} \tag{3.4.4}$$

whatever the size of $\boldsymbol{A}$.

Now consider a forest with minimum polynomial $f_1(t)$ of degree k, and let $\boldsymbol{f}_1$ be the $(k+1)$-vector $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, 1)^T$ of *coefficients in reversed order*. Then for every $(k+1)$-vector $\boldsymbol{a}$ in every sequence of the forest, we have

$$\boldsymbol{f}_1^T \boldsymbol{a} = 0. \tag{3.4.5}$$

We say that $\boldsymbol{a}$ is *annihilated* by $\boldsymbol{f}_1$.

Denote by $\boldsymbol{S}$ a matrix whose columns consist of all the $(k+1)$-vectors annihilated by $\boldsymbol{f}_1$ and no others. Then

$$\boldsymbol{f}_1^T \boldsymbol{S} = 0. \tag{3.4.6}$$

With each vector $\boldsymbol{a}$ in the forest, we can associate a $(k+1)$-copse in the forest with corresponding vectors $\boldsymbol{b}$, $\boldsymbol{c}$.

Now
$$\begin{aligned} 0 = \boldsymbol{f}_1^T \boldsymbol{S} &= \boldsymbol{f}_1^T \boldsymbol{A}^3 \boldsymbol{S} \\ &= (\boldsymbol{f}_1^T \boldsymbol{A})(\boldsymbol{A}^2 \boldsymbol{S}) \\ &= (\boldsymbol{f}_1^T \boldsymbol{A}^2)(\boldsymbol{A}\boldsymbol{S}). \end{aligned} \tag{3.4.7}$$

Write
$$\boldsymbol{f}_1^T \boldsymbol{A} = \boldsymbol{f}_2^T, \quad \boldsymbol{f}_1^T \boldsymbol{A}^2 = \boldsymbol{f}_3^T \tag{3.4.8}$$

giving two new polynomials $f_2(t), f_3(t)$ also of degree k. Again, we have for every $(k+1)$-vector $\boldsymbol{a}$

$$\boldsymbol{c} = \boldsymbol{A}\boldsymbol{a}, \quad \boldsymbol{b} = \boldsymbol{A}^2\boldsymbol{a}$$

so that every $\boldsymbol{c}$ is in $\boldsymbol{A}\boldsymbol{S}$, and every $\boldsymbol{b}$ in $\boldsymbol{A}^2\boldsymbol{S}$.

But
$$\boldsymbol{f}_2^T \boldsymbol{A}^2 \boldsymbol{S} = 0 \quad \text{and} \quad \boldsymbol{f}_3^T \boldsymbol{A}\boldsymbol{S} = 0, \tag{3.4.9}$$

and hence $\boldsymbol{f}_2$ *annihilates every* $(k+1)$*-vector* $\boldsymbol{b}$ and $\boldsymbol{f}_3$ *annihilates every* $(k+1)$*-vector* $\boldsymbol{c}$.

Since a forest can be constructed uniquely once every $(k+1)$-vector it may contain is known, and one k-vector is given, we can say that $f_2(t)$, given by (3.4.8) *is the minimum polynomial for the forest containing vectors* $\boldsymbol{b}$, and $f_3(t)$ *given by* (3.4.8) *is the minimum polynomial for the forest containing vectors* $\boldsymbol{c}$.

3.5. *Algebraic relations between* $f_1(t), f_2(t), f_3(t)$

If we write $\boldsymbol{t}(t)$ for the vector $(1, t, t^2, \ldots, t^k)^T$ and $\boldsymbol{t}^*(t)$ for $(t^k, t^{k-1}, \ldots, t, 1)$ we have immediately

$$f_1(t) = \boldsymbol{f}_1^T \boldsymbol{t}, \quad f_1^*(t) = \boldsymbol{f}_1^T \boldsymbol{t}^* \tag{3.5.1}$$

whence $$f_2(t) = \boldsymbol{f}_2^T \boldsymbol{t}(t) = \boldsymbol{f}_1^T \boldsymbol{A}\boldsymbol{t}(t) = \boldsymbol{f}_1^T \boldsymbol{t}^*(t+1)$$

$$= f_1^*(t+1) = (t+1)^k f_1\left(\frac{1}{t+1}\right). \tag{3.5.2}$$

Likewise $$f_3(t) = (t+1)^k f_2\left(\frac{1}{t+1}\right) = t^k f_1\left(1+\frac{1}{t}\right), \tag{3.5.3}$$

giving simple algebraic relations between $f_1(t)$, $f_2(t)$ and $f_3(t)$.

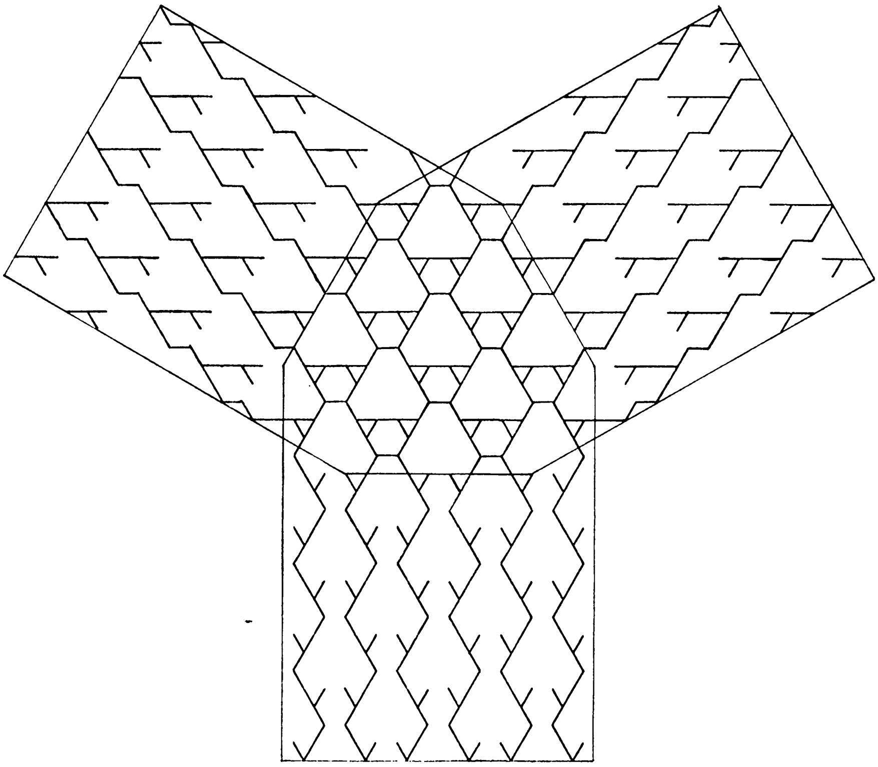

FIGURE 8. Three forests and a tessellation with a single background.

3.6. *Example*

As an illustration consider $f_1(t) = t^4+t^3+t^2+t+1$

the minimum polynomial for the forest with $n_1 = 5$, $m_1 = 3$, exhibited as the bottom forest in figure 8. For an origin at the bottom left corner, the value of $\phi_0^*(t)$ is $t+t^2+t^3$; for $\phi_0^*(t) = 1$, the 'origin' would be at [1, 2]. It is readily seen that the periods along, e.g. $r+s = 15$, and $r = 0$ are $n_2 = n_3 = 15$, and that, in each case, all parallel sloping rows are translates of the same sequence, so that $m_2 = m_3 = 1$, and $n_1 m_1 = n_2 m_2 = n_3 m_3 = 15$. A cell, clearly seen in the centre tessellation of the figure, consists of 30 triangles, 15 of each kind.

The corresponding forests, also illustrated in figure 8, have minimum polynomials $f_2(t)$ and $f_3(t)$. With $\boldsymbol{f}_1 = (1, 1, 1, 1, 1)^T$ and $\boldsymbol{A}$ a 5×5 matrix, exhibited in (3.4.2) if we stop at the 5th row up, we find $\boldsymbol{f}_2^T = \boldsymbol{f}_1^T \boldsymbol{A} = (1, 0, 0, 1, 1)^T$, so that

$$f_2^*(t) = t^4+t+1 = f_1(t+1).$$

Written in our usual convention

$$f_2^*(t) = 1+t+t^4 \quad \text{and} \quad f_3^*(t) = 1+t^3+t^4.$$

For the sequence (in the f_1-forest) generated by $f_2(t)$ along $r+s = 15$, starting with $s = 0$ we find $\phi_0^*(t) = t+t^2+t^3$. For the 'bottom' line of the section (turn through 120°) exhibiting the f_2-forest, we have $\phi_0^*(t) = 1+t^2$, starting from the left-corner (which is live).

4. Development of the theory

4.1. *Separation into subforests: alternation*

Consider now
$$G_{s+2}(t) = (1+t^{-1})^2 G_s(t)$$
$$= (1+t^{-2})\, G_s(t) \tag{4.1.1}$$

which yields
$$w_{r,\,s+2} = w_{r,\,s} + w_{r+2,\,s} \tag{4.1.2}$$

whence coefficients in G_{s+2} for r even depend only on those in G_s for r even; likewise those in G_{s+2} for r odd depend only on those in G_s for r odd. Thus the set of alternate rows G_{s+2k}, $k = 0, 1, 2, \ldots$, breaks up into two completely independent subforests. Similarly, the set of intermediate rows, which also alternate in the original array, yield two further subforests which again are independent of one another. The whole array thus breaks up into four subforests, as already described in §2.4.

The sequence of nodes in any line parallel to the chosen ground has a period n, where n is the period of that ground line in the original forest. This need not be the least period. In fact, if n is even, then $\frac{1}{2}n$ is a period for the corresponding line in every subforest, and one or more may have least period that is a proper submultiple of n (n odd) or $\frac{1}{2}n$ (n even). In fact, one subforest may be the zero forest, in which case the other three are identical.

The idea of alternation clearly extends, since

$$G_{s+2^j}(t) = (1+t^{-2^j})\, G_s(t) \tag{4.1.3}$$

so that
$$w_{r,\,s+2^j} = w_{r,\,s} + w_{r+2^j,\,s} \tag{4.1.4}$$

for all r, s.

It follows immediately from this that if the original period is $n = 2^j$, any power of 2, then $w_{r,\,s+2^j} = 0$ for all r, and that:

All trees terminate after at most 2^j rows if the base period is 2^j.

It also follows that:

Every purely periodic forest of even period may be built up from four separate forests of half the period. Two of these forests are independent, the other two derived from these. By repeating this process, every purely periodic forest may be built up from subforests of odd base period.

Alternation applied to a forest with odd periods $n \times m$ always yields four identical forests of the same periods $n \times m$ (cf. §4.6), for clearly alternate repetitions of a particular node along a base line in any of the three possible directions appear alternately in two subforests.

Examples of alternation. Figure 28, with $n = 14$, when alternation is applied, yields three copies of figure 13 ($n = 7$) at double scale, and one zero forest, call it F0. Likewise figure 24 ($n = 12$) yields three of figure 11 ($n = 6$) and one F0, while figure 11 in turn yields three of figure 9 ($n = 3$) and one F0. Figure 23 (also $n = 12$), however, yields four copies of figure 11. See table 6 for a list of Fn; see also §4.6.

4.2. *Separation into subforests with irreducible $f^*(t)$: addition of forests*

We have seen in §2.3 that forests may be 'added' to give another forest, while each forest is determined by its root-cycle, in turn given by its generating fraction $\phi_0^*(t)/f^*(t)$. Selmer (1966, ch. 3) gives the proof that such fractions may be added in a similar fashion to give sums of sequences that correspond. Likewise any such fraction may be separated into component partial fractions, in which all the denominators are powers of polynomials irreducible in GF(2). It is also known that any minimal generating polynomial $f^*(t)$ involving any factor to a power higher than the first yields a sequence of even period, and that only such generating polynomials can do so.

The consideration of §§4.1 and 4.2 combine to yield:

All forests that are purely periodic may be separated into subforests of odd base-period.

Any generating fraction $\phi_0^*(t)/f^*(t)$ that yields a sequence of odd period n may be separated into partial generating fractions, each with denominator irreducible in GF(2).

4.3. *Purely periodic forests and non-periodic trunks*

Consider a forest F' in which row-periodicity applies only for $s \geqslant s_0$. Once periodicity is established we can define a purely periodic forest F which agrees with F' for $s \geqslant s_0$, but differs for $0 \leqslant s < s_0$.

We can add these forests to give $F+F'$ (mod 2), which is a forest that *terminates at the row* $s = s_0-1$. The part of F' that differs from F is called a *non-periodic trunk* or simply *trunk* and we now show that it is due to the addition to F of a terminating forest generated by $\phi^*(t)/(1+t)^{s_0}$ for some $\phi^*(t)$ with degree $< s_0$.

Selmer (1966, Th. VI.5, p. 82, due to Morgan Ward) shows that if $2^{j-1} < s_0 \leqslant 2^j$, the period of sequences generated by $(1+t)^{s_0}$ is 2^j; these yield terminating forests, as we have seen. It also follows from (3.3.5) that if $f^*(t)$ contains any irreducible factor other than $(1+t)$, say $\lambda(t)$, then the forest cannot vanish, for $(1+t)^s\phi_0^*(t) = 0 \bmod f^*(t)$ implies $(1+t)^s = 0 \bmod f^*(t)$; however, this is ruled out since Galois field theory demonstrates the existence of an integer T, for any such $\lambda(t)$, such that $(1+t)^T = 1 \bmod \lambda(t)$.

Thus, *a terminating forest is generated only by some power of* $(1+t)$. The trunk has s_0 rows, for

$$\frac{(1+t)^{s_0-1}\phi^*(t)}{(1+t)^{s_0}} = \frac{\phi^*(t)}{1+t} = \frac{1}{1+t}$$

since $(\phi^*(t), 1+t) = 1$. This fraction generates the sequence (1, 1, 1, ...), which terminates at that line.

In a purely periodic forest of period n we thus have (using $f^\alpha \| g$ to mean $f^\alpha | g, f^{\alpha+1} \nmid g$) if

$$(1+t) \nmid f^*(t), \quad (1+t)^\alpha \| (1+t^n) = F^*(t) f^*(t)$$

so that $$(1+t)^\alpha \| F^*(t) \quad \text{and} \quad (1+t)^\alpha | \sigma_s^*(t), \quad \text{all } s. \tag{4.3.1}$$

We note that $\alpha = 2^j$, a power of 2, always, for if $n = 2^j q$, q odd, then

$$(1+t^n) = (1+t^{2^j q}) = (1+t^q)^{2^j} \quad \bmod 2$$

and for q odd, $(1+t) \| (1+t^q)$.

Hence $$(1+t)^{2^j} \| (1+t^n). \tag{4.3.2}$$

4.4. *Clearings*

The usefulness of clearings has been mentioned. We now investigate their structure a little more closely. We have denoted the generating polynomial of the first root-cycle by $\sigma_0^*(t)$; a particular forest has mn possible polynomials $\sigma^*(t)$ (using a general notation without suffix) from which it may be derived, since we may start with any of n different nodes in m different rows. We seek polynomials $\sigma^*(t)$ which have as low a degree as possible. It is evident that these correspond to sequences that contain the longest row of vacant nodes in the largest clearing (or one of these if there are more than one), and finish with as many zero coefficients as possible, i.e. the sequence starts at the live node just after the longest run of vacant nodes, and finishes with the end of the next repetition of that run of vacant nodes. If the clearing is of size k, $\sigma^*(t)$ will have degree $n-k-1$ in t. This defines a $\sigma_0^*(t)$ to have as low a degree as possible.

Consider now $\sigma_{-1}^*(t)$, the polynomial from the previous row from which $\sigma_0^*(t)$ has been derived. Since $(1+t)\,|\,\sigma^*(t)$ in any periodic forest, we can obtain $\sigma_{-1}^{*\prime}(t)$ from

$$t^{-1}\sigma_{-1}^{*\prime}(t) = \sigma_0^*(t)/(1+t) \tag{4.4.1}$$

without modification modulo $(1+t^n)$. This is of degree $n-k-2$, and must therefore have $k+1$ successive vacant nodes in the cycle; this is not permissible for this clearing. Thus

$$t^{-1}\sigma_{-1}^*(t) = (\sigma_0^*(t)+1+t^n)/(1+t). \tag{4.4.2}$$

Now $(1+t)^\alpha\,|\,\sigma_{-1}^*(t)$, whence $(1+t)^{\alpha+1}\,|\,(\sigma_0^*(t)+1+t^n)$. But $(1+t)^\alpha\,\|\,(1+t^n)$ and we must also have $(1+t)^\alpha\,\|\,\sigma_0^*(t)$.

Thus:

For the first line of any clearing, with $\sigma^*(t)$ defined from a sequence starting within the clearing,

$$(1+t)^\alpha\,\|\,\sigma^*(t) \tag{4.4.3}$$

where $(1+t)^\alpha\,\|\,(1+t^n)$, $\alpha = 2^j\|n$.

Clearings are thus characterized by having a first line with a $\sigma^*(t)$ that is exactly divisible by $(1+t)^\alpha = (1+t)^{2^j}$, where $2^j\|n$. This is a useful property for the identification of distinct forests in terms of the largest clearings they contain (see Miller 1968).

4.5. *Row-periods*

For a purely periodic forest of base-period n, we have defined various corresponding row-periods, m, S, T. We now derive a bound M, such that $m|M$, that depends only on n.

If $n = 2^jq$, q odd, then $q\,|\,(2^e+1)$ or $q\,|\,(2^e-1)$ for some least e, and

$$(1+t^n) \equiv (1+t^q)^{2^j}\,|\,(1+t^{2^e\pm 1})^{2^j} = (1+t^{2^{e+j}\pm 2^j})$$

so that
$$t^{2^{e+j}} = t^{\mp 2^j} \mod (1+t^n). \tag{4.5.1}$$

Now, if $s = 2^{e+j}$, we have

$$\left.\begin{aligned} t^s\sigma_s^*(t) &= (1+t)^{2^{e+j}}\sigma_0^*(t) \\ &= (1+t^{2^{e+j}})\,\sigma_0^*(t) \\ &= (1+t^{\mp 2^j})\,\sigma_0^*(t) \\ &= \eta'(1+t)^{2^j}\,\sigma_0^*(t) \\ &= \eta\sigma_{s'}^*(t) \end{aligned}\right\} \mod (1+t^n) \tag{4.5.2}$$

in which $\eta' = 1$ or t^{-2^j}, $s' = 2^j$ and $\eta = t^{2^j}$ or 1.

Thus there is a period $M = s-s' = (2^e-1)\,2^j$. This is clearly a symmetric period, like S or T, for it is unchanged if $\sigma_0^*(t)$ is reversed; it need not be the least S-period, and the R-period may be shorter still. It is never the T-period if $q|(2^e+1)$, but may be if $q|(2^e-1)$. For most forests of given $n > n_0$, this period is the shortest R-period, m.

We can now see that if $(1+t)^{2^j}|\sigma_0^*(t)$, the forest is purely periodic, for we may use

$$s = 2^{e+j}-2^j$$

in (4.5.2) and finish with $s' = 0$.

To summarize (including (4.3.1)):

If the base-period $n = 2^jq$, q *odd*, $q|(2^e\pm 1)$, *e least, then the row-period* $m|S$ and $S|\,2^j(2^e-1) = M$. *The forest is purely periodic if and only if* $(1+t)^{2^j}|\sigma_s^*(t)$ *for all s.*

Periods shorter than M can arise as follows. We have expressed a root sequence in the form $\phi_0^*(t)/f^*(t)$, and the sequence can be developed in ascending powers of t by using long division, in GF (2), of $\phi_0^*(t)$ by $f^*(t)$. Eventually the remainder $\phi_0^*(t)\times t^n$ is met again after n steps. If, in the meantime, the remainder $(1+t)\,\phi_0^*(t)\times t^{n-\rho-1}$ is encountered, this is

$$t\phi_1^*(t)\times t^{n-\rho-1} \bmod (1+t^n).$$

In this case $m = 1$ and $\sigma_0^*(t)$, $\sigma_1^*(t)$ give translates of the same sequence, which match if $\sigma_0^*(t)$ starts at $r = 0$, $\sigma_1^*(t)$ starts at $r = \rho$. This occurs, for example, with $f^*(t) = 1+t+t^4$, see §3.6 and figure 5.

Again we may find, mod $(1+t^n)$, a remainder $(1+t)^\beta\,\sigma_0^*(t)\times t^{n-\rho-\beta}$, which is

$$t^\beta\sigma_\beta^*(t)\times t^{n-\rho-\beta}.$$

If β_0 is the least such β, then $m = \beta_0$ is the row-period, and $\sigma_0^*(t)$, $\sigma_\beta^*(t)$ give translates of the same sequence.

Examples where $n|(2^e+1)$, $m|(2^e-1)$ are given in figures 10 (5×3), 15, 16, 17 (all 9×7), 18, 19, 20 (all 10×6), 21, 22 (both 11×31), and 27 (13×63). Other cases, where n, m both divide 2^e-1, are mentioned elsewhere in the paper.

4.6. *Periods under alternation*

Each subforest obtained by alternation is determined by a root-cycle, obtained by alternation of the original root-cycle, and we can study each subforest separately.

If the base period is even, then it is halved by alternation. We already know, from §4.5, that if n is odd, then m is also odd, and both remain unaltered by alternation. We now show that if n is even, m is also even, and is also halved by alternation.

The row-period m is determined by the least m for which

$$t^\rho(1+t)^m = 1 \quad \bmod f^*(t). \tag{4.6.1}$$

When n is even we know $f^*(t)$ must contain a squared factor $(g^*(t))^2 = g^*(t^2)$ and so

$$t^\rho(1+t)^m = 1 \quad \bmod g^*(t^2). \tag{4.6.2}$$

Then m, ρ are both even, for if not, suppose $m = 2\mu+1$, $n = 2\nu$, then

$$t^\rho(1+t^2)^\mu + t^{\rho+1}(1+t^2)^\mu - 1 = 0 \quad \bmod g(t^2).$$

The modulus is an even function and so must divide *both* even and odd parts of the left side; that is both $t^\rho(1+t^2)^\mu-1$ and $t^{\rho+1}(1+t^2)^\mu$ if ρ is even, or both $t^\rho(1+t^2)^\mu$ and $t^{\rho+1}(1+t^2)^\mu-1$ if ρ is odd. This is clearly impossible since the two parts have h.c.f. unity. Hence m is even, and then ρ is even by a similar argument.

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Immediately $$t^{\frac{1}{2}\rho}(1+t)^{\frac{1}{2}m} = 1 \mod g^*(t) \tag{4.6.3}$$
follows, and we have m and ρ, as well as n, halved by alternation.

It is now immediately clear that:

In any forest, both n and m are odd multiples of the same power 2^j. Alternation, after j steps, gives a set of subforests with m, n both odd.

Further alternation yields further forests of period $n \times m$ and eventually leads to repetition after an *alternation-* or A-*period*. We do not consider this further.

The remarks in §4.1 show further that this repetition can occur only when n, m are odd, and that, at each stage in the alternation, there is no ambiguity since all subforests are identical.

Period $n = 15$ yields examples with alternation period $A = 1$—each of figures 44 and 47; $A = 2$—figures 45 and 46 alternately; $A = 4$—F 59, F 67, F 68, F 66 or F 76, F 77′, F 76′, F 77, or F 61, F 65, F 62, F 64, each set in the cyclic order given, the last having a mirror-image set as well. See table 6 for a list of Fn.

4.7. *Layers*

If $${}_1G_0(t) = \sum_0^\infty W_{r,0}\, t^r \tag{4.7.1}$$
denotes the generating function for the base-line of the forest given in the upper layer of §2.5, we can see that
$${}_1G_0(t) = (1+t^{-1}+t^{-2})\, G_0(t) \quad \mod (1+t^n) \tag{4.7.2}$$
and that the sth sequence in the lth layer is given by
$${}_lG_s(t) = (1+t^{-1}+t^{-2})^l\,(1+t^{-1})^s\, G_0(t) \quad \mod (1+t^n). \tag{4.7.3}$$
The properties of layers are related by means of the polynomial $(1+t+t^2)$ in a manner similar to that by which rows in a forest are related by the polynomial $(1+t)$.

4.8. *Examples of layers*

Applied to figure 28, a 14×2 forest, this process yields figure 29, also 14×2, which in turn yields figure 28 again. Figures 45, 46, 47, all 15×15 forests, form a cycle of layer-period 3, each yielding the next; figure 44 yields figure 45, but is outside the cycle—in fact, a *trunk*, similar to that defined in §4.3. In fact, figure 45 as a layer follows any of figures 54, 47, or forests F 75, F 75′; figure 46 follows any of figure 45 or forests F 55, F 56, F 56′; and figure 47 follows any of figure 46 or forests F 57, F 58, F 58′. Thus, in this case we have a cycle of 3, each of which, besides its cyclic predecessor, has three possible trunks, each a forest 15×15. These trunks could be extended further back, but the base period would become 45, then 135, and so on.

We shall not study further in this paper the interesting properties of layers and the layer-period, and the part this process can play in the identification and classification of distinct forests. We mention only that $f(t) = t^9+t+1$ generates an interesting set of forests with layer period 7, and $n = 73$, $m = 1$.

5. Characteristics and determination of row-periods for forests of given base-period n

5.1. *The row-period bound M*

We have seen in §4.5 that for any forest of base-period $n = 2^jq$, q odd, the row-period
$$m \mid M = 2^j(2^e-1),$$
where e is the least integer such that $q \mid (2^e \pm 1)$. We now show that $m = M$ if $\sigma_0^*(t) = (1+t)^{2^j}$.

In §4.6 we saw that $2^j \parallel n$ implies $2^j \parallel m$ and vice versa. We need therefore consider only j-fold alternations with periods n', m', M' where $n = 2^j n'$, $m = 2^j m'$, $M = 2^j M'$. In this case

$$\sigma_0^*(t) = 1+t \quad \text{and} \quad (1+t) \parallel (1+t^{n'}).$$

Thus
$$(1+t)^{m'} \sigma_0^*(t) = t^{m'} \sigma_{m'}^*(t) = t^u \sigma_0^*(t) \mod (1+t^{n'}) \tag{5.1.1}$$

so that
$$t^u(1+t) = (1+t)^{m'+1} \mod (1+t^{n'}) \tag{5.1.2}$$

for some u. This can only happen if $(1+t)^{m'+1}$ is binomial, i.e. if $m'+1 = 2^i$, a power of 2. We then have
$$t^u(1+t) = 1+t^{2^i} \mod (1+t^{n'}) \tag{5.1.3}$$

and clearly this implies
$$t^{2^i} = t^{\pm 1} \mod (1+t^{n'}) \tag{5.1.4}$$

or that
$$n' \mid (2^i \pm 1). \tag{5.1.5}$$

Now for least row period m', i must be as small as possible and so $i = e$, since $n' = q \mid (2^e \pm 1)$, with least e; $m' = 2^e - 1$ and $m = 2^j(2^e-1) = M$, as stated.

For any other forest of base-period n, we have

$$\sigma_0^*(t) = g^*(t)\,(1+t)^{2^j} \tag{5.1.6}$$

(see §4.3), for some polynomial $g^*(t)$, and m is also a row-period for this forest. It will be the least row-period if $(g^*(t), f^*(t)) = 1$, but *may* not be so otherwise. In this

$$(1+t^n) = (1+t)^{2^j} f^*(t)$$

when (5.1.6) holds, and also $(g^*(t), f^*(t)) = 1$.

To summarize:

For a base-period $n = 2^j q$, q odd, the row-period bound $M = 2^j(2^e-1)$, $(q \mid (2^e \pm 1)$, least e), is attained for $\sigma_0^*(t) = (1+t)^{2^j}$, and for any $\sigma_0^*(t) = g^*(t)\,(1+t)^{2^j}$ with $(g^*(t), f^*(t)) = 1$, when

$$1+t^n = (1+t)^{2^j} f^*(t).$$

Examples of forests with the same base-period, but different row-periods may be seen in figures 13 and 14 (both $n = 7$, but $m = 1, 7$ respectively). Likewise for $n = 14$, figures 28 and 29 have $m = 2$, while figures 30 to 41 have $m = 14$. For $n = 15$, figure 8 shows $m = 1$, figures 42 and 43 show $m = 3$, and figures 44 to 47 have $m = 15$, the maximum. Other cases are enumerated in table 5.

5.2. *Sequences and cycles of period n*

The total number of sequences of period n is 2^n, including sequences of smaller least period. For purely periodic forests, we are concerned only with sequences having a factor $(1+t)^{2^j}$, where $n = 2^j q$, q odd. There are $2^{n-\alpha} = 2^{(q-1)\alpha}$ of these 'usable' sequences, writing $2^j = \alpha$, and for later use we write

$$K(n) = 2^{(q-1)\alpha} = 2^{(q-1)2^j}. \tag{5.2.1}$$

Each sequence of period d that yields a purely periodic forest also yields a similar forest of period n if $d \mid n$. If we denote by $N(n)$ the number of sequences yielding periodic forests of least period n, clearly

$$K(n) = \sum_{d \mid n} N(d) \tag{5.2.2}$$

whence, by the Möbius inversion formula (Hardy & Wright 1960, p. 236)

$$N(n) = \sum_{d \mid n} \mu(d)\, K\left(\frac{n}{d}\right). \tag{5.2.3}$$

This yields the values listed in table 3 to $n = 50$, and also the corresponding numbers $C(n)$ of distinct cycles, given by

$$N(n) = nC(n). \tag{5.2.4}$$

It is also of interest to count reflexive forests, having vertical axes of symmetry. Each cycle in sequences used in such forests has two distinguishable centres of reflexion; these centres may be

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at a node, or *half-way between* two nodes. If n is odd the cycle has one centre of each type, if n is even it has two of the same type.

In a reflexive forest, each axis of reflexion runs *through* and *between* nodes on crossing alternate rows. There are thus equally many centres of symmetry of the two kinds included in one complete set of distinct cycles of the forest. It is useful to note that all nodes *on* lines of symmetry must be vacant, for a live branch cannot reach such a node symmetrically.

We can now count suitable cycles. We need only count symmetry centres of one kind, since there are equally many of each. This number is also the number of suitable cycles; if n is odd, each cycle contains just one of these centres; if n is even, half the cycles contain two of them, the others none. We count those having a vacant node at the symmetry centre. There are then $2^{\frac{1}{2}(n-1)}$ such centres if n is odd, $2^{\frac{1}{2}n-1}$ if n is even.

If $n = 2^j q$, q odd, $j > 1$, we do not want all these cycles, however, but only those for which $(1+t)^{2^j}$ divides the corresponding $\sigma^*(t)$. It is perhaps simplest to count these by forming a basis for the space of vectors giving usable cycles, a subspace of the space of n-vectors corresponding to the set of all symmetric n-cycles. Taking the centre of symmetry as origin of index, the basis-vectors may be taken as

$$(t+t^{-1})^{2^{j-1}}(t+t^{-1})^i \qquad i = 0, 1, 2, \ldots, \tfrac{1}{2}(n-2^j)-1.$$

Thus the number $\kappa(n)$ of suitable symmetric cycles is

$$\kappa(n) = 2^{\frac{1}{2}(n-2^j)} \quad \text{where} \quad 2^j \,\|\, n. \tag{5.2.5}$$

We then have that the number $\Sigma(n)$ of symmetric cycles of exact period n, usable in purely periodic forests is

$$\Sigma(n) = \sum_{d\mid n} \mu(d)\,\kappa\left(\frac{n}{d}\right) \tag{5.2.6}$$

in which $\kappa(r) = 2^{\frac{1}{2}(r-\alpha_r)}$ where $\alpha_r = 2^{j_r} \,\|\, r$.

Table 3 lists numbers of such cycles.

Table 3. Numbers of usable sequences and cycles of least period n

n	$N(n)$	$C(n)$	$\Sigma(n)$
3	3	1	1
5	15	3	3
6	12	2	2
7	63	9	7
9	252	28	14
10	240	24	12
11	1023	93	31
12	240	20	12
13	4095	315	63
14	4032	288	56
15	16365	1091	123
17	65535	3855	255
18	65268	3626	238
19	2 62143	13797	511
20	65280	3264	240
21	10 48509	49929	1015
22	10 47552	47616	992
23	41 94303	1 82361	2047
24	65280	2720	240
25	167 77200	6 71088	4092
26	167 73120	6 45120	4032
27	671 08608	24 85504	8176
28	167 73120	5 99040	4032
29	2684 35455	92 56395	16383
30	2684 18820	89 47294	16242
31	10737 41823	346 36833	32767
33	42949 66269	1301 50493	65503
34	42949 01760	1263 20640	65280
35	1 71798 69105	4908 53403	1 31061
36	42949 01520	1193 02820	65268
37	6 87194 76735	18572 83155	2 62143
38	6 87192 14592	18084 00384	2 61632
39	27 48779 02845	70481 51355	5 24223
40	42949 01760	1073 72544	65280
41	109 95116 27775	2 68173 56775	10 48575
42	109 95105 75156	2 61788 23218	10 47494
43	439 80465 11103	10 22801 51421	20 97151
44	109 95105 79200	2 49888 76800	10 47552
45	1759 21860 27780	39 09374 67284	41 94162
46	1759 21818 50112	38 24387 35872	41 92256
47	7036 87441 77663	149 72073 22929	83 88607
48	42949 01760	894 77120	65280
49	28147 49767 10592	574 43872 79808	167 77208
50	28147 49599 33200	562 94991 98664	167 73108

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TABLE 4. IRREDUCIBLE POLYNOMIALS (SEE §4.6)

k	polynomial $f(t)$	n	m	ρ	$2D$	S	T
2	111	3	1	1		1	3
3	1011	7	1	4	9	7	7
4	11111	5	3	1		3	15
	11001	15	1	3	7	15	15
5	100101	31	1	13	27	31	31
	111101	31	1	11	23	31	31
	111011	31	1	18	37	31	31
6	1001001	9	7	1		7	63
	1011011	63	1	7	15	21	63
	1010111	21	3	10	23	63	63
	1110011	63	1	24	49	9	63
	1100001	63	1	5	11	63	63
7	10000011	127	1	120	241	127	127
	10101011	127	1	106	213	127	127
	10111111	127	1	108	217	127	127
	10001001	127	1	96	193	127	127
	11101111	127	1	72	145	127	127
	10001111	127	1	40	81	127	127
	10011101	127	1	9	19	127	127
	11010011	127	1	38	77	127	127
	10100111	127	1	13	27	127	127
8	111010111	17	5	6		5	85
	110111101	85	1	15	31	85	85
	100111001	17	15	1		15	255
	100101101	255	1	15	31	255	255
	100011011	51	5	10	25	255	255
	111011101	85	3	50	103	255	255
	101110001	255	1	24	49	255	255
	110011111	51	5	31	67	255	255
	110001011	85	3	35	73	255	255
	111001111	255	1	114	229	255	255
	111111001	85	3	18	39	255	255
	100101011	255	1	12	25	51	255
	111000011	255	1	98	197	255	255
	101001101	255	1	232	465	17	255
	101100011	255	1	58	117	85	255
	111110101	255	1	121	243	85	255
9	1000000011	73	1	64	129	73	73
	1000010111	73	7	44	95	511	511
	1011001111	511	1	70	141	511	511
	1010100011	511	1	475	951	511	511
	1001001011	73	7	4	15	511	511
	1001011001	511	1	371	743	511	511
	1101101011	511	1	125	251	511	511
	1001100101	73	7	60	121	511	511
	1111100011	511	1	427	855	511	511
	1000011011	511	1	314	629	511	511
	1000010001	511	1	381	763	73	511
	1100110001	511	1	103	207	511	511
	1100100011	511	1	113	227	511	511
	1001011111	511	1	458	917	73	511
	1010010101	511	1	285	571	511	511
	1111001011	511	1	134	269	511	511

Table 4. Irreducible polynomials (*cont.*)

k	polynomial $f(t)$	n	m	ρ	$2D$	S	T
9	1011011011	511	1	10	21	73	511
	1001101111	511	1	418	837	511	511
	1010110111	511	1	50	101	511	511
	1100111011	511	1	262	525	73	511
	1001110111	511	1	68	137	511	511
	1011010001	511	1	274	549	511	511
	1001111101	511	1	277	555	511	511
	1110000101	511	1	193	387	511	511
	1111111011	511	1	403	807	511	511
	1010101111	511	1	58	117	511	511
	1010111101	511	1	485	971	511	511
	1110001111	511	1	184	369	511	511
10	11111111111	11	31	1		31	341
	11110000111	341	1	31	63	341	341
	10010101001	33	31	1		31	1023
	10111000111	1023	1	31	63	341	1023
	11000100011	33	31	1		31	1023
	11000110111	1023	1	31	63	341	1023
	10000110101	93	11	79	169	1023	1023
	10001100011	341	3	11	25	1023	1023
	11000100101	1023	1	219	439	1023	1023
	11100101011	93	11	13	37	1023	1023
	11011001101	341	3	297	597	1023	1023
	10011100111	1023	1	549	1099	1023	1023
	10100001011	93	11	53	117	341	1023
	11011111101	1023	1	847	1695	341	1023
	10010000001	1023	1	76	153	341	1023
	10000011101	341	1	179	359	341	341
	10010101111	341	1	305	611	341	341
	11111000101	341	1	39	79	341	341
	11010111111	341	3	138	279	33	1023
	11110010011	1023	1	711	1423	1023	1023
	11001011011	1023	1	125	251	1023	1023
	11011110111	341	3	323	649	93	1023
	11101111101	1023	1	273	547	1023	1023
	10010001011	1023	1	56	113	1023	1023
	10001000111	341	3	278	559	1023	1023
	11100010111	1023	1	324	649	93	1023
	11101010101	1023	1	746	1493	1023	1023
	10110111001	341	3	223	449	1023	1023
	10110010111	1023	1	258	517	93	1023
	11101011001	1023	1	944	1889	1023	1023
	10000001111	341	3	331	665	1023	1023
	11111111001	1023	1	585	1171	1023	1023
	10110000101	1023	1	920	1841	1023	1023
	10001010011	341	3	23	49	1023	1023
	11001000011	1023	1	354	709	1023	1023
	11000010101	1023	1	755	1511	1023	1023
	10010011001	341	3	230	463	1023	1023
	10110100001	1023	1	180	361	1023	1023
	10100111101	1023	1	893	1787	1023	1023
	10101100111	341	3	204	411	1023	1023
	11100100001	1023	1	84	169	1023	1023
	10100110001	1023	1	686	1373	1023	1023

TABLE 4. IRREDUCIBLE POLYNOMIALS (*cont.*)

k	polynomial *f*(*t*)	*n*	*m*	*ρ*	2*D*	*S*	*T*
10	10110101011	341	3	207	417	1023	1023
	11011000001	1023	1	492	985	1023	1023
	10100011001	1023	1	83	167	1023	1023
	10001101111	1023	1	448	897	341	1023
	11111011011	1023	1	622	1245	341	1023
	11010110101	1023	1	289	579	341	1023
	10011110011	1023	1	79	159	341	1023
	11001111111	1023	1	601	1203	341	1023
	11110001101	1023	1	763	1527	341	1023

k	polynomial *f*(*t*)	*n*	*m*	*ρ*	2*D*	*S*	*T*
11	101011100011	23	89	5	7	2047	2047
	100011000011	89	23	16	55	2047	2047
	100100110111	89	23	63	149	2047	2047
	100111101111	89	23	59	141	2047	2047
	110111111111	89	23	73	169	2047	2047
12	1111111111111	13	63	1		63	819
	1011110010111	35	117	25	27	4095	4095
	1011110001111	39	35	34	25	1365	1365
	1000000001001	45	91	26	53	4095	4095
	1010011100101	65	21	22		21	1365
	1000111110001	65	63	1		63	4095
	1011101011101	65	63	1		63	4095
	1101011101011	65	63	1		63	4095
	1000101111001	91	45	76	15	4095	4095
	1011000000011	91	45	3	51	4095	4095
	1101001111011	91	45	26	97	4095	4095
14	100111111111001	43	127	1		127	5461
	101010010010101	43	127	1		127	5461
	110100010001011	43	127	1		127	5461
18	1111111111111111111	19	511	1		511	9709
	1000000001000000001	27	511	1		511	13797
	1011100001000011101	57	511	1		511	29127
	1111011101011101111	57	511	1		511	29127
20	100001000010000100001	25	1023	1		1023	25575
	101111100111001111101	41	341	14		341	13981
	110110100111001011011	41	1023	1		1023	41943
	101101101001011100111	55	6355	34	43	349525	349525
	100000000000000100001	75	13981	56	143	1048575	1048575
21	1000000100000000000001	49	42799	23	19	2097151	2097151
22	10100110011000001100111	69	60787	11	89	4194303	4194303
23	100011000111011011101111	47	178481	10	89	8388607	8388607

Table 5. Enumeration of forests (see §6.3)

n	m (factors of t^n+1 and identification)	number of forests: all	number of forests: R-forests	generating polynomials
3	11×111			
	a			
	1	1	1	a
5	11×11111			
	a			
	3	1	1	a
6	$11^2 \times 111^2$			
	a			
	2	1	1	a^2
7	$11 \times 1011 \times 1101$			
	a a'			
	1	2	—	a, a'
	7	1	1	aa'
9	$11 \times 111 \times 1001001$			
	a b			
	7	4	2	b, ab
10	$11^2 \times 11111^2$			
	a			
	6	4	2	a^2
11	11×11111111111			
	a			
	31	3	1	a
12	$11^4 \times 111^4$			
	a			
	4	5	3	a^3, a^4
13	11×1111111111111			
	a			
	63	5	1	a
14	$11^2 \times 1011^2 \times 1101^2$			
	a a'			
	2	4	—	a^2, a'^2
	14	20	4	$a^2a', aa'^2, a^2a'^2$
15	$11 \times 111 \times 11111 \times 10011 \times 11001$			
	a b c c'			
	1	2	—	c, c'
	3	3	1	ab, ac, ac'
	15	72	8	$cc', acc', bc, abc, bc', abc', bcc', abcc'$
17	$11 \times 100111001 \times 111010111$			
	a b			
	5	3	3	a
	15	256	16	b, ab
18	$11^2 \times 111^2 \times 1001001^2$			
	a b			
	14	259	17	b^2, a^2b, ab^2, a^2b^2
19	$11 \times 1111111111111111111$			
	a			
	511	27	1	a
20	$11^4 \times 11111^4$			
	a			
	12	272	20	a^3, a^4

TABLE 5. ENUMERATION OF FORESTS (*cont.*)

	factors of t^n+1 and identification			
		number of forests		
n	m	all	R-forests	generating polynomials
21	$11\times111\times1011\times1101\times1010111\times1110101$			
	a b b' c c'			
	1	2	—	ab, ab'
	3	2	—	c, c'
	7	1	1	abb'
	9	2	—	ac, ac'
	21	18	—	bc, bc', $b'c$, $b'c'$, $bb'c$, $bb'c'$
	63	786	16	multiples of cc'; or with a *and* b or b' *and* c or c'
22	$11^2\times11111111111^2$			
	a			
	62	768	16	a^2
23	$11\times101011100011\times110001110101$			
	a a'			
	89	2	—	a, a'
	2047	89	1	aa'
24	$11^8\times111^8$			
	a			
	8	340	30	a^5, a^6, a^7, a^8
25	$11\times11111\times100001000010000100001$			
	a b			
	1023	656	4	b, ab
26	$11^2\times1111111111111^2$			
	a			
	126	5120	32	a^2
27	$11\times111\times1001001\times1000000001000000001$			
	a b c			
	511	4864	16	c, ac, bc, abc
28	$11^4\times1011^4\times1101^4$			
	a a'			
	4	72	—	a^3, a'^3, a^4, a'^4
	28	21384	144	multiples of a^3a' or aa'^3
29	$11\times11111111111111111111111111111$			
	a			
	16383	565	1	a
30	$11^2\times111^2\times11111^2\times10011^2\times11001^2$			
	a b c c'			
	2	8	—	c^2, c'^2
	6	63	7	multiples of ab, ac, ac', at least one squared
	30	298230	540	others with c or c' and at least one squared factor
31	$11\times100101\times101001\times101111\times111101\times110111\times111011$			
	a a' b b' c c'			
	1	6	—	any single polynomial
	31	1117317	1057	any product
33	$11\times111\times11111111111\times10010101001\times11000100011$			
	a b c d			
	31	4198403	2113	any multiple of c, d, or ab
34	$11^2\times100111001^2\times111010111^2$			
	a b			
	10	192	24	a^2
	30	4210624	2168	b^2, ab^2, a^2b, a^2b^2

Table 5. Enumeration of forests (*cont.*)

n	*m* (factors of t^n+1 and identification)	all (number of forests)	R-forests (number of forests)	generating polynomials
35	11 × 1011 × 1101 × 11111 × 1011110010111 × 1110100111101			
	a *a′* *b* *c* *c′*			
	3	2	—	*ab*, *a′b*
	21	1	1	*aa′b*
	117	2	—	*c*, *c′*
	585	6	—	*bc*, *bc′*
	819	18	—	*ac*, *ac′*, *a′c*, *a′c′*, *aa′c*, *aa′c′*
	4095	119862	32	multiples of *cc′*; or with *a* or *a′* and *b* and *c* or *c′*
36	$11^4 \times 111^4 \times 1001001^4$			
	a *b*			
	28	4260815	2331	multiples of a^3b, b^3
37	11 × 111111111111111111111111111111111111			
	a			
	87381	21255	3	*a*
38	$11^2 \times 1111111111111111111^2$			
	a			
	1022	1769472	256	a^2
39	11 × 111 × 1111111111111 × 1011110001111 × 1111000111101			
	a *b* *c* *c′*			
	35	6	—	*c*, *c′*
	63	5	1	*ab*
	105	6	—	*ac*, *ac′*
	1365	1260	6	*cc′*, *acc′*
	4095	1720740	126	multiples of *bc*, *bc′*
40	$11^8 \times 11111^8$			
	a			
	24	4473856	2720	a^5, a^6, a^7, a^8
41	11 × 1011111001110011111101 × 1101101001110010111011			
	a *b*			
	341	75	3	*a*
	1023	26214400	1024	*b*, *ab*
42	$11^2 \times 111^2 \times 1011^2 \times 1101^2 \times 1010111^2 \times 1110101^2$			
	a *b* *b′* *c* *c′*			
	2	22	—	*ab*, *ab′*, at least one squared
	6	32	—	c^2, c'^2
	14	101	13	*abb′*, at least one squared
	18	162	—	*ac*, *ac′* at least one squared
	42	19008	—	*bc*, *bc′*, *b′c*, *b′c′*, *bb′c′*, *bb′c′* at least one squared
	126	207762066	8312	other multiples of *cc′*, or of *a and b* or *b′ and c* or *c′*, with a square
43	11 × 110100010001011 × 101010010010101 × 100111111111001			
	a *b* *c*			
	127	805355523	16513	any
44	$11^4 \times 11111111111^4$			
	a			
	124	201523200	8448	a^3, a^4
45	11 × 111 × 11111 × 1001001 × 10011 × 11001 × 1000000001001 × 1001000000001			
	a *b* *c* *d* *d′* *e* *e′*			
	21	12	2	*bc*, *abc*, *cd*, *acd*, *cd′*, *acd′*
	91	32	—	*e*, *e′*, *de*, *d′e′* (*not de′*, *d′e*)
	105	216	8	multiples of *bcd*, *bcd′*, *cdd′*, (without *e* or *e′*)
	273	32	—	*ae*, *ae′*, *ade*, *ad′e′*
	819	896	—	*ce*, *ce′*, *ace*, *ace′*, *cde*, *cd′e′*, *acde*, *acd′e′*
	1365	2176	—	any other factor of *abdd′e*, *abdd′e′* with *e* or *e′*
	4095	95466112	1024	multiples of *ee′*, *bce*, *bce′*, *cde′* or *cd′e*

Table 5. Enumeration of forests (*cont.*)

n	factors of t^n+1 and identification: m	number of forests: all	number of forests: R-forests	generating polynomials
46	$11^2 \times 101011100011^2 \times 110001110101^2$			
	a a'			
	178	1024	—	a^2, b^2
	4094	93414400	1024	a^2b, ab^2, a^2b^2
47	$11 \times 100011000111011011101111 \times 111101110110111000110001$			
	a a'			
	178481	2	—	a, a'
	8388607	178481	1	aa'
48	$11^{16} \times 111^{16}$			
	a			
	16	5592320	4080	$a^{9(1)16}$
49	$11 \times 1011 \times 1101 \times 1000000100000000000001 \times 1000000000000010000001$			
	a a' b b'			
	42799	16	—	b, b', ab', $a'b$
	299593	16	—	ab, $a'b'_1$, $aa'b$, $aa'b'$
	2097151	2739136	8	multiples of bb'
50	$11^2 \times 11111^2 \times 100001000010000100001^2$			
	a b			
	2046	2751465884	8198	b^2, ab^2, a^2b, a^2b^2

5.3. *The determination of row-periods:* (i) *general remarks*

We first recall that the sequence given by $\phi_0^*(t)/f^*(t)$ has the same period as that given by $1/f^*(t)$ when $(\phi_0^*(t), f^*(t)) = 1$. The row-period of the corresponding forest is also the same, for this is given by the least m for which

$$t^{-\rho}\phi_0^*(t) = t^m\phi_m^*(t) = (1+t)^m\phi_0^*(t) \quad \bmod f^*(t)$$

for some translation ρ, and if $(\phi_0^*(t), f^*(t)) = 1$ this implies (see (4.6.1))

$$t^\rho(1+t)^m = 1 \quad \bmod f^*(t) \tag{5.3.1}$$

so that the least m is independent of $\phi_0^*(t)$.

Thus the base-period and row-period of the purely periodic forest generated by $\phi_0^*(t)/f^*(t)$, a fraction in its lowest terms, depends only on $f^*(t)$.

Again, the fraction $\phi_0^*(t)/f^*(t)$ can be split into partial fractions with denominators that are powers of irreducible polynomials. For any such fraction with $f_i^*(t) = \{p^*(t)\}^r$, in which $p^*(t)$ gives periods $n \times m$, the periods for $f_i^*(t)$ are $2^j n \times 2^j m$, where j is given by $2^{j-1} < r \leqslant 2^j$.

We therefore need to determine periods only for fractions $1/p^*(t)$, where $p^*(t)$ is irreducible, and to build up other periods from these. For base-periods this is simply a matter of least common multiples. For row-periods we need also the *phase*, or amount of translation, ρ, of the repetition at row m.

5.4. *The determination of row-periods:* (ii) $f^*(t)$ *irreducible*

Suppose now $f^*(t) = p^*(t)$, an irreducible polynomial of degree k. There are $2^k - 1$ non-zero polynomials $\phi_0^*(t)$, of degree less than k, yielding sequences appearing in purely periodic forests, i.e. that have $\phi_s^*(t)/f^*(t)$ in lowest terms; all have the same base period n, such that $n|(2^k-1)$, and the same row-period m, such that $m|(2^k-1)$. The number of distinct forests is in fact $(2^k-1)/mn$.

We have seen also that if k is even, and if $n|(2^{\frac{1}{2}k}+1)$, then $m|(2^{\frac{1}{2}k}-1)$; here $\frac{1}{2}k = e$ of §4.5, in other cases $k = e$.

Selmer (1966) gives information about the period n, and methods for its determination, though there remains the need for a considerable amount of elementary computing for complete analysis of periods in particular cases (e.g. by long division, or streamlined versions of this for use by hand or automatic computer).

Likewise the determination of row-periods m remains largely a matter of trial for individual polynomials, again with the possibility of various short cuts.

As an illustration of some possibilities, consider the case $n = 17$, which exhibits a property of some row-periods that has not yet received full explanation. We have

$$t^{17}+1 = 11 \times 100111001 \times 111010111.$$

The last two factors are of degree 8, and might be expected to yield identical periods. In each case there are 255 polynomials $\phi_s^*(t)$, that is, there are 15 cycles of period 17. Since $17 = 2^4+1$, the maximum row-period is $2^4-1 = 15$. We find for $p^*(t) = 100111001$, that, in fact, $m = 15$, and that the cycles for this denominator are arranged as *one* forest 17×15. However, for

$$p^*(t) = 111010111$$

we find, unexpectedly, that $m = 5$, and that the corresponding cycles are arranged in *three* forests 17×5. No simple criterion has been found to distinguish the difference in behaviour of such apparently similar polynomials, and we determine m by trial in each case.

Evaluation by trial can be carried out by systematic use of relation $(1+t)^{2^j} = 1+t^{2^j} \pmod 2$. The row-period bound is known for given n, either $(2^k-1)/n$, or $2^{\frac{1}{2}k}-1$ if $n|2^{\frac{1}{2}k}+1$. The least row-period must be a submultiple of the appropriate one of these. Let d be any divisor, being tested as a possible period. We have to see whether

$$t^{\rho}(1+t)^d = 1 \quad \operatorname{mod} p^*(t) \tag{5.4.1}$$

for some ρ. We test instead, whether

$$t^{\rho}(1+t)^d F^*(t) = F^*(t) \quad \operatorname{mod}(1+t^n) \tag{5.4.2}$$

where $p^*(t)\, F^*(t) = (1+t^n)$; this must be done for a selection of d.

If the bound M^* for d is $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_l^{\alpha_l}$, we must test each $d = M^*/p_i^{\beta_i}$ to find the greatest β_i such that (5.4.1) is satisfied while it is not satisfied for $M^*/p_i^{\beta_i+1}$. Then

$$m = M^*/\Pi p_i^{\beta_i} \quad (0 \leqslant \beta_i \leqslant \alpha_i).$$

Some or all of the β_i are zero in the majority of cases.

To perform a test, we need $F^*(t)\,(1+t)^d \operatorname{mod}(1+t^n)$. To compute this we write

$$d = \gamma_1+\gamma_2+\gamma_3+\dots, \quad \gamma_i = 2^{j_i}.$$

Then
$$\begin{aligned} F^*(t)\,(1+t)^d &= F^*(t)\,(1+t)^{\gamma_1}(1+t)^{\gamma_2}(1+t)^{\gamma_3}\dots \\ &= F^*(t)\,(1+t^{\gamma_1})\,(1+t^{\gamma_2})\,(1+t^{\gamma_3})\dots \quad \operatorname{mod}(1+t^n) \end{aligned}$$

where multiplication by each factor is a binomial cyclic multiplication mod $(1+t^n)$. It is useful also to work from d_r to d_{r+1}, i.e. from one d to another, by expressing $d_{r+1}-d_r$ in binary form similarly; this can provide useful checks if we always finish a run by *satisfying* (5.4.2). It is useful also to obtain explicit expressions for $(1+t)^r F^*(t)$ for $r = 1, 2, 3, \ldots, 16$, say, in order to be able to use these to check (5.4.2) in the form:

$$t^{\rho}(1+t)^{d+r} F^*(t) = (1+t)^r F^*(t).$$

In this way, one can reduce the number of binomial multiplications on occasion.

5.5. *Example: the case* $n = 41$

As an illustration consider $n = 41$.

$$t^{41}+1 = 11 \times A \times B = 11 \times 101111100111001111101 \times 110110100111001011011.$$

Since $41 | (2^{10}+1)$, we have $M = 2^{10}-1 = 1023 = 3 \times 11 \times 31$. For the large factors, $k = 20$, and there are 1048575 sequences or 25575 cycles. The smaller possibilities for m, i.e. 1, 3, 11, 31, 33 may be eliminated by constructing the first 33 rows of the forest; this leaves 93, 341, 1023 as possibilities.

For the first factor, A, $F^*(t) = 11 \times B$; this is the first line of the calculation below:

$n = 41$ $p^*(t) = 1$ 01111 10011 10011 11101

$\sigma_0(t) = F^*(t)$	·····	·····	·····	····1	·11·1	11·1·	·1·11	1·11·	1
$\times t^{-18} \equiv t^{64}$	·1·11	·111·	1··1·	111·1	1·1··	·····	·····	·····	·
$t^{64}\sigma^*_{64}(t)$	·1·11	·111·	1··1·	111··	11··1	11·1·	·1·11	1·11·	1
$\times t^{-9} \equiv t^{32}$	·1··1	·111·	·11··	111·1	··1·1	11·11	·1·1·	11·11	1
$t^{96}\sigma^*_{96}(t)$	···1·	·····	1111·	····1	111··	····1	····1	·11·1	·
$\times t^{5} \equiv t^{128}$	11·1·	···1·	·····	1111·	····1	111··	····1	····1	·
$t^{224}\sigma^*_{224}(t)$	11···	···1·	1111·	11111	111·1	111·1	·····	·11··	·
$\times t^{-18} \equiv t^{64}$	11111	·1111	·1···	···11	···11	·····	·1·11	11·11	1
$t^{288}\sigma^*_{288}(t)$	··111	·11·1	1·11·	111··	1111·	111·1	·1·11	1·111	1
$\times t^{-9} \equiv t^{32}$	11·11	·111·	·1111	·111·	1·1·1	11·11	11··1	11·11	·
$t^{320}\sigma^*_{320}(t)$	111··	···11	11··1	1··1·	·1·11	··11·	1··1·	·11··	1
$\times t^{16}$	··11·	1··1·	·11··	1111·	····1	111··	11··1	··1·1	1
$t^{336}\sigma^*_{336}(t)$	11·1·	1···1	1·1·1	·11··	·1·1·	11·1·	·1·11	·1··1	·
$\times t^{4}$	··1·1	1·1·1	···11	·1·1·	11···	1·1·1	1·1··	1·11·	1
$t^{340}\sigma^*_{340}(t)$	11111	··1··	1·11·	··11·	1··1·	·1111	11111	11111	1
$t^{341}\sigma^*_{341}(t)$	·····	1·11·	111·1	··1·1	11·11	·1···	·····	·····	·

Instead of using $d_1 = 93$, we use $d'_1 = 96$, and compare with $(1+t)^3 F^*(t)$ (not here recorded). Thus $d'_1 = 2^6+2^5$. Now $2^6 = 64 = -18 \pmod{41}$, so we repeat the first line translated *back* 18 places; we then add, and use $t^{32} \equiv t^{-9}$ on the sum by translating back 9 places and adding again. This yields $t^{96}\sigma^*_{96}(t)$, which is *not* of form $t^{-\rho}\sigma^*_3(t)$, as is immediately obvious by the absence of a gap of 16 consecutive vacant nodes.

Next $\qquad d_2 - d'_1 = 341 - 96 = 245 = 2^7+2^6+2^5+2^4+2^2+1,$

which we shall use this time (we could have used 256). In the last line, when multiplying by $1+t$, this has been done directly, without rewriting the previous line. The result reproduces the first line, displaced 14 places to the left. Thus $m = 341$, $\rho = 14$; from this $2D = m+2\rho = 41 \pmod{82}$, and we have an S-period, expected as A is symmetric.

For the factor B, we find $m = 1023$, also an S-period.

This information suffices to fix the size and number of all forests for $n = 41$. For the factor A as denominator there are

$$(2^{20}-1)/41 = 25575 \text{ cycles}$$

yielding **75 forests 41 × 341.**

For both other denominators B and AB, the row-period is 1023. Thus for B there are

25 forests 41 × 1023.

For $n = 41$ there are $2^{40}-1$ sequences, of these $2^{20}-1$ have generator A, and $2^{20}-1$ have generator B. Thus there are

$$2^{40}-2.2^{20}+1 = 109\ 95095\ 30625 \text{ sequences}$$

giving 2 68173 05625 cycles

whence **262 14375 forests 41 × 1023**

with denominator AB.

5.6. *Relations between forests with a common background* (*table* 4)

It is useful now to list relations between constants for purely periodic forests sharing a common background.

Suppose $f_1(t), f_2(t), f_3(t)$ are the generating polynomials for the three aspects. Then we have, see (3.4.8),

$$f_3^T = f_2^T A = f_1^T A^2 \tag{5.6.1}$$

which shows, incidentally that $f_1(t), f_2(t), f_3(t)$ have the same degree since $1+t \nmid f_i(t)$. They must also factorize into corresponding irreducible factors of the same degree for each forest, since to any factorization there corresponds a separation into constituent forests by partial fractions, and this separation applies to the background *as a whole*. Equality of degree then applies to the generating polynomials of each constituent forest. Thus all of $f_1(t), f_2(t), f_3(t)$ are simultaneously reducible, or simultaneously irreducible.

We need here to consider reciprocal polynomials also as primary generating polynomials. For this we may use the notation $f'(t)$ in place of $f^*(t)$ so that, for example, $f^{*\prime}(t) = f(t)$ as a function of t, but with coefficients *written* in reverse order (see §3.2).

Diagram 4 shows points of repetition, A and C, of sequences starting at O. The point A is as near as possible to the r axis, and C as near as possible to the s axis; the parallelograms given by OA, OR or by OC, OS represent minimum cells of the background; here OR, OS are periods along the n-, s-axes. To $f_1(t)$ correspond periods n_1, m_1 and displacements ρ_1, D_1; the displacements ρ_1', D_1' correspond to $f_1'(t)$. Similarly, m_3, ρ_3' etc. correspond to $f_3'(t)$.

Since A, C are congruent to O, and are as near Or, Os as possible, we have m_1, m_3 as least periods and ρ_1, ρ_3' periods with

$$\rho_1 = k_3 m_3, \quad \rho_3' = k_1' m_1. \tag{5.6.2}$$

Also $m_1 | n_3$, since n_3 is a row-period for $f_1(t)$.

Thus, using (see §2.6)

$$m_1 n_1 = m_2 n_2 = m_3 n_3 = C \tag{5.6.3}$$

we have and likewise

$$\left.\begin{aligned} m_3 &= (\rho_1, n_1) \qquad & n_3 &= C/m_3 = m_1 n_1/(\rho_1, n_1) \\ m_2 &= (\rho_1', n_1) \qquad & n_2 &= C/m_2 \end{aligned}\right\} \tag{5.6.4}$$

where

$$\rho_1 + \rho_1' + m_1 = 0 \quad \mod n_1. \tag{5.6.5}$$

Next ρ_1, ρ_3' satisfy the relations

$$\left.\begin{aligned} t^{\rho_1}(1+t)^{m_1} &= 1 \\ t^{m_3}(1+t)^{\rho_3'} &= 1 \end{aligned}\right\} \quad \mathrm{mod}\, f_1^*(t) \tag{5.6.6}$$

whence, using (5.6.2) to eliminate $(1+t)$, we obtain

$$t^{\rho_1 k_1'} = t^{m_3} \quad \mathrm{mod}\, f_1^*(t)$$

and, again using (5.6.2), $\qquad (k_3 k_1' - 1)\, m_3 = 0 \quad \mathrm{mod}\, n_1$

or $\qquad k_3 k_1' = 1 \ \mathrm{mod}\, (n_1/m_3 = C/m_1 m_3)$

and likewise $\qquad k_2' k_1 = 1 \quad \mathrm{mod}\, (C/m_1 m_2). \qquad (5.6.7)$

From these ρ_3', ρ_2 are readily computed.

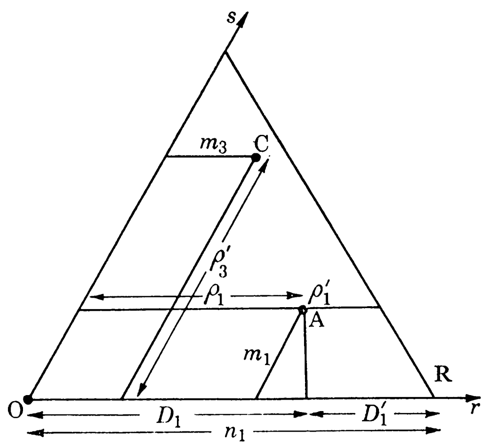

Diagram 4

It is sometimes useful to write (5.6.6) in a more symmetrical form

$$\left.\begin{aligned} t^{D_i}(t^{-\frac{1}{2}}+t^{\frac{1}{2}})^{m_i} &= 0 \quad \mathrm{mod}\, f_i^*(t) \\ 2D_i &= m_i + 2\rho_i \end{aligned}\right\} \tag{5.6.8}$$

with

in which $2D_i$ is an odd integer and is always equal to n_i (mod $2n_i$) for an S-period. Then (5.6.5) becomes

$$D_i + D_i' = n_i \ (\mathrm{mod}\ 2n_i). \tag{5.6.5$'$}$$

Here D_1 represents the displacement of A in x, while ρ_1 represents the displacement in r.

Finally we have

$$\left.\begin{aligned} S_i &= C/(2D_i, n_i), \\ T &= \mathrm{LCM}(n_1, n_2, n_3). \end{aligned}\right\} \tag{5.6.9}$$

Table 4 gives periods for irreducible polynomials to degree 10, grouped in sets of three having a common background. Periods are also given for other irreducible polynomials with $k \leqslant 23$ and $n < 100$. The values of ρ and $2D$ are also recorded, except that $2D = n$ is *omitted* in order to mark S-periods that are also m-periods.

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5.7. *The determination of row-periods:* (iii) *reducible* $f^*(t)$

Consider now the generating fraction $\phi^*(t)/f^*(t)$ where $f^*(t)$ is reducible. This may be decomposed into a sum of fractions†

$$\frac{\phi^*(t)}{f^*(t)} = \Sigma \frac{\phi_r^*(t)}{\{p_r^*(t)\}^{\alpha_r}}, \tag{5.7.1}$$

in which denominators are powers of irreducible polynomials. Each separate fraction yields a forest; these may be combined by addition.

We take first the case where $f^*(t)$ contains no squared factor. Then the characteristics of each forest given by $\phi_r^*(t)/p_r^*(t)$ can be taken from table 4, or evaluated as described earlier. When constituent forests are added it is clear that:

(i) The base-period n is the least common multiple of the base-periods of the constituents.

(ii) The S- and T-periods are the respective least common multiples of the S- and T-periods of the constituents.

(iii) The row-period is *at least* the least common multiple of the row-periods of the constituents.

We must consider the row-period more closely. Clearly repetition of the root-line in the combined forest cannot occur until the constituent forests have *simultaneously* reached repetitions of their own root-lines. However, more is needed; the repetitions must all be at the *same relative phase*, as measured by the x- or r-displacement relative to the combined root-line. We remark that no use is made in this argument of the irreducibility of denominators, hence we may without loss of generality suppose that forests are combined in pairs until the final forest is obtained.

Consider then two forests given by $\phi_r^*(t)/f_r^*(t)$ with periods, etc. given by

$$(n_r, m_r, \rho_r, 2D_r, S_r, T_r),\dagger \quad r = 1, 2$$

and we suppose $(f_1^*(t), f_2^*(t)) = 1$. For the combined forest, we have

$$\left.\begin{array}{c} n = n_1 n_2/(n_1, n_2) \\ S = S_1 S_2/(S_1, S_2) \qquad T = T_1 T_2/(T_1, T_2) \\ m \geqslant m_1 m_2/(m_1, m_2) = \mu, \text{ say} \qquad \mu \leqslant m \leqslant S \leqslant T. \end{array}\right\} \tag{5.7.2}$$

For the constituent forests, the cycles will be in phase after μ rows if we can find ρ so that

$$\mu_2 \rho_1 \,(\text{mod}\, n_1) \equiv \rho \equiv \mu_1 \rho_2 \,(\text{mod}\, n_2) \tag{5.7.3}$$

where $$m_1 = \mu_1(m_1, m_2) \qquad m_2 = \mu_2(m_1, m_2)$$

so that $$\mu = \mu_1 \mu_2 (m_1, m_2). \tag{5.7.4}$$

If this equation cannot be solved (the usual case) we must find the least κ for which we can solve

$$\kappa\mu_2\rho_1 \,(\text{mod}\; n_1) \equiv \rho \equiv \kappa\mu_1\rho_2 \,(\text{mod}\; n_2). \tag{5.7.5}$$

This will give $m = \kappa\mu$ as the row period. We need, then,

$$\rho = \kappa\mu_2\rho_1 + \lambda_1 n_1 = \kappa\mu_1\rho_2 + \lambda_2 n_2$$

whence $$\kappa(\mu_2\rho_1 - \mu_1\rho_2) = \lambda_2 n_2 - \lambda_1 n_1. \tag{5.7.6}$$

For this to have a solution, (n_1, n_2) must divide $\kappa(\mu_2\rho_1 - \mu_1\rho_2)$; we must choose the least κ for which this is so. Thus

$$\kappa = (n_1, n_2)/(\mu_2\rho_1 - \mu_1\rho_2, n_1, n_2) \tag{5.7.7}$$

† Note that the suffix r now refers to distinct constituent forests, and not to rows in a forest, nor to the triangular aspect.

and
$$m = m_1 m_2 (n_1, n_2)/(m_1, m_2)\,(\mu_2\rho_1 - \mu_1\rho_2, n_1, n_2)$$
$$= m_1 m_2 (n_1, n_2)/(m_2\rho_1 - m_1\rho_2, (n_1, n_2)\,(m_1, m_2)) \qquad (5.7.8)$$
and ρ is then readily derived.

If denominators to powers higher than the first occur, then both m and n are multiplied by the same power 2^j, given by $2^{j-1} < \alpha \leqslant 2^j$, where α is the highest power to which any repeated factor in $f^*(t)$ occurs, and the same combination principles apply.

The method works equally well if $2D$ is used in place of ρ.

5.8. *Example: determination of row-periods for $n = 45$*

When $n = 45$, we may factorize to obtain
$$t^{45}+1 = 11 \times 111 \times 11111 \times 1001001 \times 10011 \times 11001 \times 1000000001001 \times 1001000000001.$$
We extract from table 4 the following details:

label	factor	k	n	m	ρ
a	111	2	3	1	1
b	11111	4	5	3	1
c	1001001	6	9	7	1
d	10011	4	15	1	11
d'	11001	4	15	1	3
e	1000000001001	12	45	91	26
e'	1001000000001	12	45	91	18

This covers all irreducible factors. Here the label is used to identify factors easily; k is the degree of the factor.

Forests with period $n = 45$ must have at least one factor e or e', or a factor c combined with one or more of b, d or d'.

Consider $f^*(t) = bc$, $\qquad n = 5 \times 9/(5, 9) = 45$
$$m = 3 \times 7/(3, 7) = 21$$
since b and c are both symmetric.

Next take $f^*(t) = cd$, $\qquad n = 9 \times 15/(9, 15) = 45$
$$m \geqslant 7 \times 1/(7, 1) = 7.$$
At the 7th row, $\qquad \mu_2 = 1, \quad \rho_1 = 1 \pmod 9$
$$\mu_1 = 7, \quad 7\rho_2 = 7 \times 11 \pmod{15} = 2 \pmod{15}$$
and we need to solve $\qquad \kappa(1-2) = 15\lambda_2 - 9\lambda_1$

whence $\kappa = 3$, and $m = 7\kappa = 21$ as for bc. Also
$$\rho = 3 \pmod 9 = \kappa\rho_1 \pmod 9$$
$$= 231 \pmod{15} = 7\kappa\rho_2 \pmod{15}$$
$$= 21 \pmod{45}.$$

To summarize, for $f^*(t)$ not involving e, e', we find:

Periods 45×21 for bc, abc, cd, acd, cd', acd'.

Periods 45×105 for bcd, $abcd$, bcd', $abcd'$, cdd', $acdd'$, $bcdd'$, $abcdd'$.

As an illustration with row-period 105, take $f^*(t) = bcd$.

For cd $\qquad n_1 = 45, \quad m_1 = 21, \quad \rho_1 = 21.$

For b $\qquad n_2 = 5, \quad m_2 = 3, \quad \rho_2 = 1.$

For bcd $\qquad n = 45, \quad m \geqslant 21, \quad \mu = 21, \quad \mu_1 = 7, \quad \mu_2 = 1.$

We must consider, at the 21st row

$$\rho_1 = 21 \pmod{45}$$

and
$$7\rho_2 = 7 \pmod 5$$

and solve
$$\kappa(21-7) = 5\lambda_2 - 45\lambda_1.$$

Clearly $\kappa = 5$, whence $m = 105$ and

$$\begin{aligned}\rho &= \kappa\mu_2\rho_1 \pmod{45} = 105 \pmod{45}\\ &= \kappa\mu_1\rho_2 \pmod 5 = 0 \pmod 5\\ &= 15 \pmod{45}.\end{aligned}$$

For generating polynomials involving e or e', we must have odd multiples of 91 as row-period. Consider de, yielding $n = 45$, $m \geqslant 91$ in the usual way. Also

$$\begin{aligned}\rho &= \kappa\times 91\times 11 \pmod{15} = 11\kappa \pmod{15}\\ &= \kappa\times 1\times 26 \pmod{45} = 26\kappa \pmod{45}\end{aligned}$$

so that $\kappa = 1$, $m = 91$, and $\rho = 26$.

For de', again $n = 45$, $m \geqslant 91$, but this time

$$\begin{aligned}\rho &= \kappa\times 91\times 11 \pmod{15} = 11\kappa \pmod{15}\\ &= \kappa\times 1\times 18 \pmod{45} = 18\kappa \pmod{45}\end{aligned}$$

so that $\kappa = 15$, $m = 1365$ and $\rho = 0$, $\rho' = 30$ (for $d'e$).

In this way we find the periods as follows, in each case with $n = 45$.

$m = 91$ $\quad f^*(t) = e, e', de, d'e'$

$m = 273$ $\quad f^*(t) = ae, ae', ade, ad'e'$

$m = 819$ $\quad f^*(t) = ce, ce', ace, ace', cde, cd'e', acde, acd'e'$

$m = 1365$ $\quad f^*(t) =$ any other factor of $abdd'e$, $abdd'e'$ with e or e'

$m = 4095$ $\quad f^*(t) =$ any multiple of ee', bce, bce', cde' or $cd'e$.

6. Enumeration of forests of given base-period n

6.1. *Forests generated by irreducible* $f^*(t)$

The number of forests in this case is simply found. If the degree of $f^*(t) = p^*(t)$ is k, the base-period n, and the row-period m, then all forests have same periods $m\times n$, and their number is

$$N(p^*(t)) = (2^k-1)/mn.$$

Thus $f^*(t) = 111010111$, of degree 8 yields *one* forest 17×15, and $f^*(t) = 100111001$ yields *three* forests 17×5.

6.2. *Forests generated by reducible* $f^*(t)$

In this case, we must count cycles generated by $f^*(t)$ and remove all those generated by proper submultiples of $f^*(t)$. To do this we need only to know the degrees of the various factors. The enumeration may be done by means of the Möbius inversion formula, as in §5.2, or sometimes more conveniently by direct appeal to the method of exclusions. We shall use the case $n = 45$ as an example to illustrate the processes involved.

6.3. *Example: enumeration of forests for* $n = 45$ (*table* 5)

Details of degrees and periods of the various polynomial factors of $t^{45}+1$ are given in §5.8.

Consider $f^*(t) = acd$; the combined degree is 12, so that there are $2^{12} = 4096$ numerators $\phi^*(t)$, including zero, each giving a separate sequence, of period 45 or a submultiple. We wish now to remove sequences generated by submultiples of *acd*; these correspond to numerators that are multiples of the cofactor, there are 2^{10} such multiples of a, 2^6 of c and 2^8 of d. We must restore multiples of ac, ad, cd, removed too often, and so on. Eventually,

$$2^{12}-2^{10}-2^6-2^8+2^2+2^6+2^4-1 = 2835 \text{ sequences}$$

remain, each of period 45, with denominator *acd* in lowest terms. There are thus 63 cycles and, since $m = 21$ for this denominator, just *three* forests 45×21.

We need not, however, consider each polynomial separately. The direct count of all sequences for each denominator includes all those with any proper divisor as denominator. We need only exclude those yielding forests of a different size, because either m or n is different.

Thus bc, abc, cd, acd, cd', acd' all yield 45×21 forests. We therefore count sequences for abc, acd, acd', which includes all the 45×21 cases, remove those for ad, ad', ab, ac (the last *three* times, since it has been included 3 times), and finally restore a again, three times. This yields

$$\begin{aligned} 3\times 2^{12}-3\times 2^6-3\times 2^8+3\times 2^2 &= 11340 \text{ sequences}\\ &= 252 \text{ cycles}\\ &= \mathbf{12\ forests\ 45\times 21.} \end{aligned}$$

Likewise, for row-period 105, using the notation (ab) to denote total number of sequences with denominator ab, we find

$$\begin{aligned} (abcdd')-(abdd')-(acd)-(acd')+(ad)+(ad')-(abc)+2(ac)+(ab)-2(a)&\\ = 1020600 \text{ sequences}&\\ = 22680 \text{ cycles}&\\ = \mathbf{216\ forests\ 45\times 105.}& \end{aligned}$$

Full results are given in table 5, containing an enumeration of all forests with $n \leqslant 50$. The table also gives factors of t^n+1, with an identifier for each factor, except $t+1$, an accent being used to indicate a reciprocal factor. The numbers of reflexive or R-forests, see §7, are also given. Generating polynomials for each possible period-combination $n\times m$ are indicated, using identifiers for the factors.

7. Reflexive forests

7.1. *Symmetric forests and tessellations*

Strictly the only proper symmetry a forest can have is reflexive symmetry. A forest is reflexive, or an R-forest, if there exist lines in which reflexion leaves the forest unaltered.

There are, however, cases where the background yields the *same* forest from all three aspects. The corresponding tessellation then has rotational symmetry, and we may say also that the forest has this symmetry.

Symmetry of tessellations will be discussed more fully in a subsequent paper. A brief account has appeared in Miller (1968); we note a few of the possibilities here.

A tessellation will be called *Reflexive* or an R-tessellation if there are lines in which reflexion leaves it unaltered.

A tessellation will be called *Rotational* if there are centres of symmetry about which rotation through $\pm 120°$ leaves it unaltered. It will be called a *Triangular* or T-tessellation if it is also reflexive, it will be called *Skew-symmetric* or an S-tessellation otherwise.

A tessellation with neither of these symmetries will be called *Unsymmetric* or a U-tessellation.

Rotational backgrounds give just one forest, three times; there is only one generating polynomial $f(t)$, which satisfies

$$\boldsymbol{f} = \boldsymbol{f}\boldsymbol{A} \quad \text{or} \quad f(t) = f^*(1+t).$$

7.2. *Enumeration of reflexive forests*

The enumeration of reflexive forests (including R-forests, and T-forests counted once only) follows closely the lines of the general enumeration of § 6, but using only symmetric cycles, and thus only symmetric numerators and denominators need to be counted. The total number of usable cycles with centre of symmetry at a node and exact period n is given by (5.2.6), and there are equally many usable cycles with symmetry between two nodes. Each cycle is represented twice in the total count, once for each symmetry centre.

It will be easiest to demonstrate enumeration with a particular example.

7.3. *Example: reflexive forests with* $n = 45$

For R-forests, dd' must occur as a combination, and so must ee'. Numerators are also symmetric.

For $m = 21$, $f^*(t)$ is either bc or abc. Now abc has degree 12, and there are 2^7 symmetric numerators *up to* degree 11 (counting the null sequence twice, it has two distinguishable symmetry centres); thus there are 2^6 symmetric cycles. We now remove those with ab, ac as denominator, and restore a again. This gives

$$2^6 - 2^4 - 2^3 + 2 = 42 \text{ cycles}$$
$$= \mathbf{2\ T\text{- and } R\text{-forests } 45 \times 21.}$$

For $m = 105$, we have denominators cdd', $bcdd'$, $acdd'$, $abcdd'$ which (using, for example, $\langle abc \rangle$ for the number of symmetric cycles with denominator abc) yields

$$\langle abcdd' \rangle - \langle abdd' \rangle - \langle abc \rangle + \langle ab \rangle$$
$$= 2^{10} - 2^7 - 2^6 + 2^3 = 840 \text{ symmetric cycles}$$
$$= \mathbf{8\ T\text{- and } R\text{-forests } 45 \times 105.}$$

The only other possible value of n is 4095 since both e and e' must occur. This gives immediately

$$\langle abcdd'ee' \rangle - \langle abcdd' \rangle = 2^{22} - 2^{10} = 4\,193\,280 \text{ S-cycles}$$
$$= \mathbf{1024\ T\text{- or } R\text{-forests } 45 \times 4095.}$$

Enumerations of reflexive forests are listed in table 5 to $n = 50$.

7.4. *Diagrams*

Table 6 lists the symmetry type of each tessellation itemized. All four types, R, S, T, U, occur together for the first time with $n = 14$. It is of interest to note here two types of T-forest, one in which close centres occur in horizontal rows, illustrated in figures 14 ($n = 7$), 32, 33 (both $n = 14$), and figures 44 to 47 (all $n = 15$). The other has close centres of symmetry (more numerous for the same n) arranged in vertical lines, we see this in figures 9 ($n = 3$), 11 ($n = 6$), 23, 24, 26 (all $n = 12$). The latter type can occur only when $3|n$, but need not (as with $n = 15$). For $n = 63$ both types occur.

We can likewise subdivide S-symmetry. The first type is illustrated by $n = 14$, figures 30, 31, 38 and all S-tessellations for $n = 15$. The second type is shown only by figure 25 ($n = 12$) in table 6; the next case has $n = 24$. There is also a third type where all lines with close centres of symmetry are skew, figures 13 ($n = 7$) and figures 21, 22 ($n = 14$) illustrate this.

8. Miscellaneous properties and problems

8.1. *Identification and listing of distinct forests* (*table* 6)

A large number of forests and tessellations have been drawn; some are included with this paper. The drawings made include all forests and tessellations for $n \leqslant 15$, all background diagrams (by computer) for $n = 17$, 18, 20, 24, rotational tessellations for $n = 21$, 24, 28, 42, 73, 85, 93, 105, and a number of others.

A problem in connexion with a complete list is to pick out the last few. One way, by systematic use of clearings has been described fully in Miller (1968); this may become clearer in conjunction with figures 28–41 for $n = 14$ in the present paper. Another method, perhaps rather more effective, and capable of extension, is to use the layered forests of § 2.6, 4.7, and extensions of this idea.

Table 6 lists individual forests with $n \leqslant 15$, labelled F 1 to F 77. About half of these are illustrated in figures 9 to 47.

8.2. *Forests with small maximum clearing size*

It is clear that there is only one forest with largest clearing of size 1. It is also easy to show that there is only one pair that have maximum clearings of size 2. The top line in diagram 5, 1001, is the only possibility for a gap 2. The *previous* line has two possibilities; one is excluded because it has gap 3, hence this line is unique. There are again two possibilities for the next predecessor but they are reflexions of one another, so we keep one only. We can immediately append a 1 to the left, to hold the gap to two. The next line is determinate, like the second, and so is the next, and the next, indefinitely. We have established a period 7, and the forest 7×1, which, with its mirror image, exhausts the possibilities for clearing of maximum size 2. See figures 3, 4, 12 and 13.

```
          1 0 0 1
         0 1 1 1 0
       1 0 0 1 0 1 1
      0 1 1 1 0 0 1 0
    1 0 0 1 0 1 1 1 0 0 1
   0 1 1 1 0 0 1 0 1 1 1 0
```

Diagram 5

On the other hand, we can construct immediately an infinite number of forests with maximum clearing of size 4. Any forest that contains a first alternation that is the 3×1 forest F 1 (figure 9), can have no larger gap than size 3 in the lines occupied by this subforest, for it contains itself gaps of at most size 1, which can have zeros on both sides of the zero in any gap, but no more; intervening lines can have gap 4—this covers all lines. On the other hand, the forest sharing the same lines as the 3×1 is completely arbitrary, and it is easy to ensure that gaps 4 *do* occur.

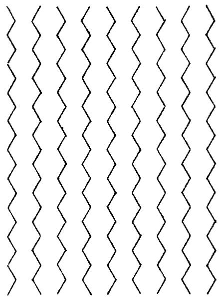

Figure 9. F1 3×1 T.

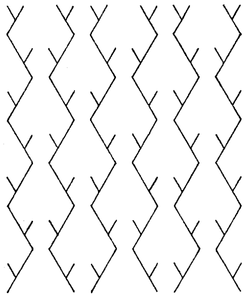

Figure 10. F2 5×3 R (see F35).

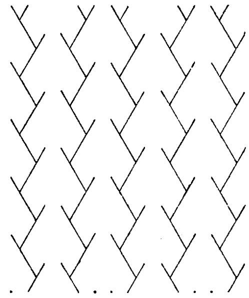

Figure 11. F3 6×2 T.

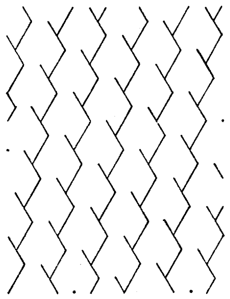

Figure 12. F4′ 7×1.

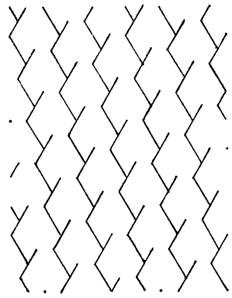

Figure 13. F4 7×1 S.

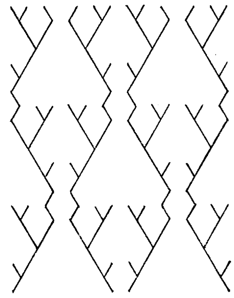

Figure 14. F5 7×7 T.

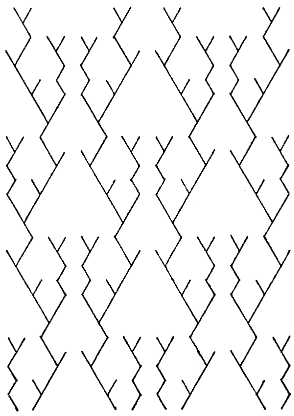

Figure 15. F6 9×7 R.

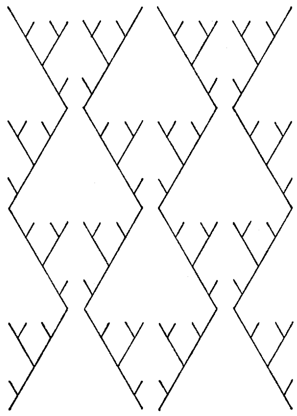

Figure 16. F7 9×7 R.

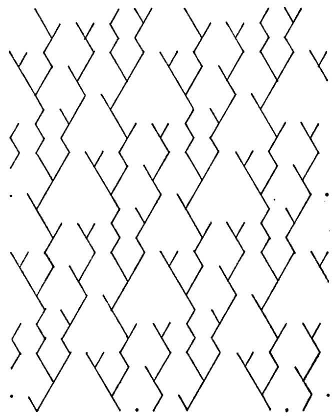

Figure 17. F8 9×7 U.

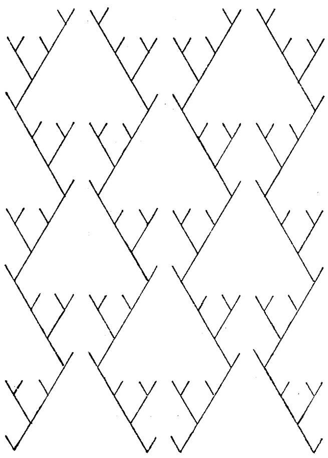

Figure 18. F9 10×6 R.

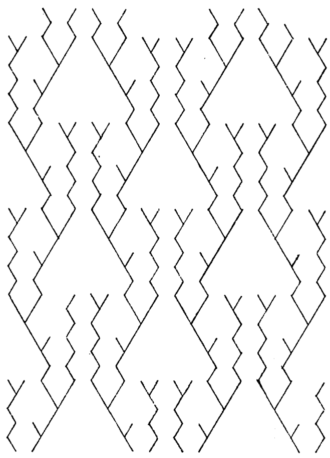

Figure 19. F10 10×6 R.

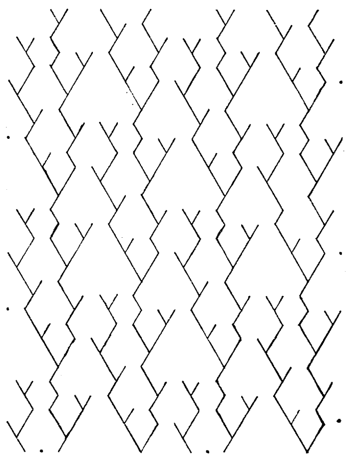

Figure 20. F11 10×6 U.

Table 6. Forests with $n \leqslant 15$ (see §8.1)

	n	m	$f(t)$	$\phi(t)$	figure	$\sigma(t)$	type
F 1	3	1	111	1	9	110	T
F 2	5	3	11111	1	10, 8	11000	R see F 35
F 3	6	2	111^2	1	11	101000	T
F 4	7	1	1011	1	13, 3	1011100	S
F 4′			1101	1	12	1110100	
F 5	7	7	1011 × 1101	1	14	1100000	T
F 6	9	7	1001001	1	15	100100000	R
F 7			111 × 1001001	1	16	110000000	R
F 8			111 × 1001001	1011	17	111010000	U
F 9	10	6	11111^2	1	18	1010000000	R
F 10			11111^2	111	19	1101100000	R
F 11			11111^2	1011	20	1001110000	U
F 12	11	31	11111111111	1	21	11000000000	R
F 13			11111111111	1011	22	11101000000	U
F 14	12	4	111^3	1	23	111011100000	T
F 15			111^4	1	24	100010000000	T
F 16			111^4	1011	25	101110110000	S
F 17			111^4	11111	26	111101111000	T
F 18	13	63	1111111111111	1	27	1100000000000	R
F 19			1111111111111	1011	*	1110100000000	U
F 20			1111111111111	10011	*	1101010000000	U
F 21	14	2	1011^2	1	28	10001010100000	S
F 22			1011^2	111	29	11101101011000	S
F 23	14	14	$1011^2 \times 1101$	1	30	11100100000000	S
F 24			$1011^2 \times 1101$	111	31	10101111000000	S
F 25			$1011^2 \times 1101^2$	1	32	10100000000000	T
F 26			$1011^2 \times 1101^2$	111	33	11011000000000	T
F 27			$1011^2 \times 1101^2$	11111	34	11000110000000	R
F 28				11001	35	11111010000000	
F 29			$1011^2 \times 1101^2$	111 × 11111	36	10010100100000	R
F 30				111 × 11001	37	10111001100000	
F 31			$1011^2 \times 1101^2$	1000011	38	10100111100000	S
F 32			$1011^2 \times 1101^2$	100101	39	10110001000000	U
F 33				111101	40	11001001000000	
F 34				111011	41	11010111000000	
F 35	15	1	11001	1	*, 8	111101011001000	see F 2
F 36	15	3	111 × 11111	1	42	101100110100000	R
F 37			111 × 11001	1	43	100111001100000	
F 38	15	15	10011 × 11001	1	*	111001110000000	R
F 39			11111 × 10011	1	*	110100010000000	
F 40			111 × 10011 × 11001	1	*	100001000000000	R
F 41			111 × 11111 × 10011	1	*	101011000000000	
F 42			111 × 10011 × 11001	1011	*	101101011000000	U
F 43			111 × 11111 × 10011	1011	*	100100101000000	
F 44			111 × 11001 × 11111	1011	*	111110111000000	
F 45			10011 × 11001 × 11111	1	45	100100000000000	T
F 46			10011 × 11001 × 11111	111	46	111111000000000	T
F 47			10011 × 11001 × 11111	111^2	47	101111010000000	T
F 48			10011 × 11001 × 11111	1011	*	101001100000000	S
F 49			10011 × 11001 × 11111	100101	*	100001101000000	S
F 50			10011 × 11001 × 11111	100011	*	100111011000000	S
F 51			10011 × 11001 × 11111	1011^2	*	100110110100000	S
F 52			10011 × 11001 × 11111	1000011	*	100101101100000	S
F 53			10011 × 11001 × 11111	10011101	*	100011101010000	S

TABLE 6. FORESTS WITH $n \leqslant 15$ (*cont.*)

	n	m	$f(t)$	$\phi(t)$	figure	$\sigma(t)$	type
F 54	15	15	111 × 10011 × 11001 × 11111	1	44	110000000000000	T
F 55			111 × 10011 × 11001 × 11111	1001001	*	110110110000000	R
F 56				1011011	*	111011010000000	
F 57			111 × 10011 × 11001 × 11111	111010111	*	100111100100000	R
F 58				110111101	*	101100011100000	
F 59			111 × 10011 × 11001 × 11111	1011	*	111010000000000	S
F 60			111 × 10011 × 11001 × 11111	100101	*	110111100000000	
F 61				111101	*	100011100000000	U
F 62				111011	*	100110100000000	
F 63			111 × 10011 × 11001 × 11111	1000011	*	110001010000000	
F 64				1100111	*	101010010000000	U
F 65				1110101	*	100111110000000	
F 66			111 × 10011 × 11001 × 11111	1011^2	*	110011110000000	S
F 67			111 × 10011 × 11001 × 11111	10000011	*	110000101000000	S
F 68			111 × 10011 × 11001 × 11111	100010001	*	110011011000000	S
F 69			111 × 10011 × 11001 × 11111	10011101	*	110100111000000	
F 70				11010011	*	101110101000000	U
F 71				10100111	*	111101001000000	
F 72			111 × 10011 × 11001 × 11111	1011 × 100101	*	111100100100000	
F 73				1011 × 111101	*	101111000100000	U
F 74				1011 × 111011	*	101011111100000	
F 75			111 × 10011 × 11001 × 11111	1001001011	*	110110111010000	S
F 76				1001101111	*	110101100010000	S
F 77				1011100111	*	111001010010000	S

* These forests have also been drawn and placed, with others, in The Royal Society Depository for Unpublished Mathematical Tables no. 88.

Examples illustrated are figure 11 ($n = 6$, combining F 1 with F 0) and figure 25 ($n = 12$, combining F 1 with F 3). Here F 0 denotes the zero forest.

What then of forests with largest clearing of size 3? This question is considered in an accompanying paper by ApSimon (1970), who shows that their number is finite.

One can extend the question a little: Are there infinitely many forests with largest gap of size 4 that do *not* include a subforest 3×1? Are there infinitely many forests with largest gap of size 5? The first question is so far unanswered; we hope to return to the latter (the answer is 'yes') in another paper.

8.3. *Tessellations of given row-period*

It is not difficult to see that reflexive tessellations of given row-period m are finite in number. Each line of symmetry has a node-period S or line period $2S$—the base-period is a symmetric one. The nodes are all vacant, but each intervening line in the tessellations may have a link between successive live nodes crossing the line of symmetry or be free of this link. There are 2^S possibilities, and a particular choice can be seen to determine a background completely.

Likewise forests or tessellations with given S-period can also be seen to be finite in number. The above argument still holds, but we may use any line parallel to the y-axis and allow any link to be present or not, but also any node on the line may be live or vacant—2^{2S} possibilities.

On the other hand, there are certainly infinitely many forests of row-period 1, for every primitive polynomial generates one. It is probable that there are infinitely many for any given m that is not an S-period. Figures 9, 13 and 8, with $n = 3$, 7 and 15 illustrate $m = 1$; $n = 21$ with $f(t) = 1010111$ gives the first case with $n \neq 2^k - 1$.

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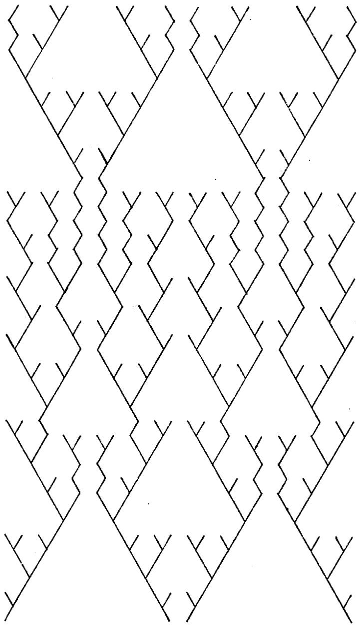

Figure 21. F 12 11 × 31 R.

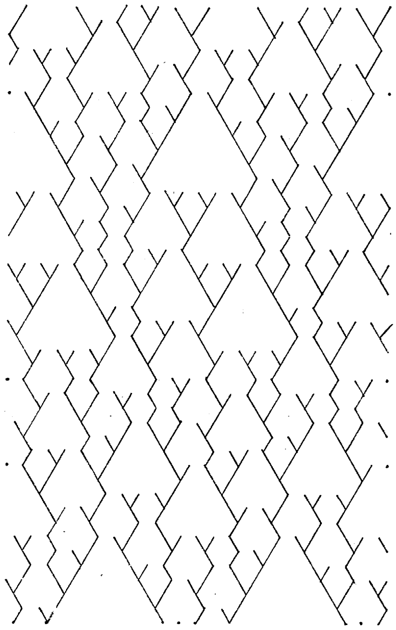

Figure 22. F 13 11 × 31 U.

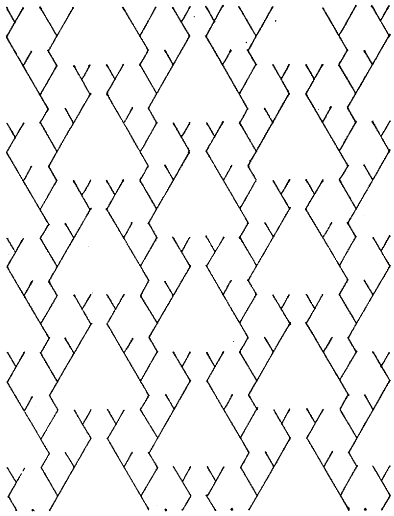

Figure 23. F 14 12 × 4 T.

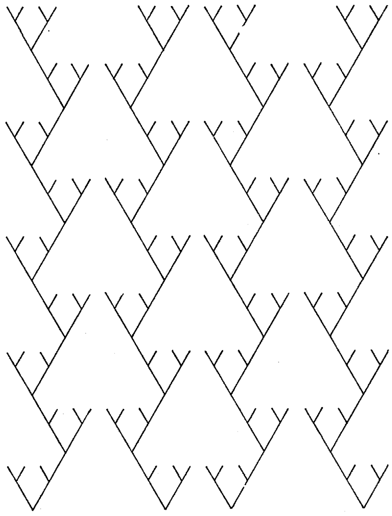

Figure 24. F 15 12 × 4 T.

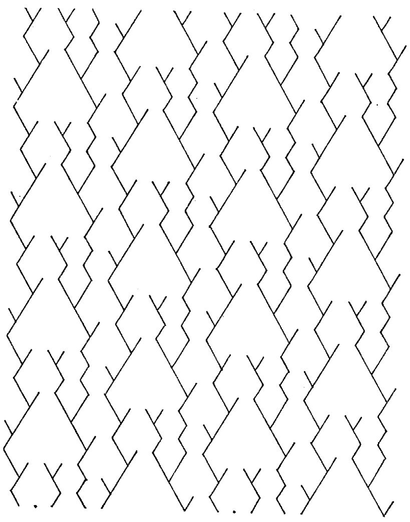

Figure 25. F 16 12 × 4 S.

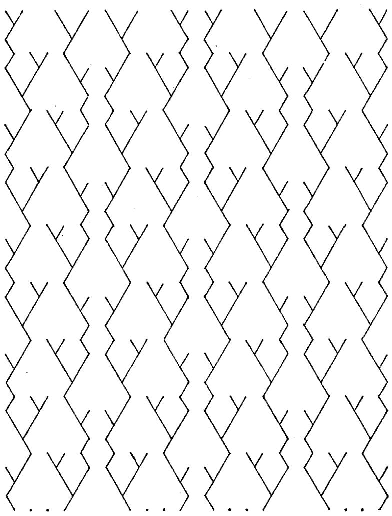

Figure 26. F 17 12 × 4 T.

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8.4. *Designing*

These patterns may be used in several ways to provide artistic designs of mathematical interest as well. Floor-tiling comes immediately to mind for the tessellations, wall-papers might well make use of the forest designs, rugs of either. As an illustration of this a colour plate is included showing two designs worked in Touch Tapestry (figures 48 and 49, plate 1), where the triangular background lends itself to this kind of design.

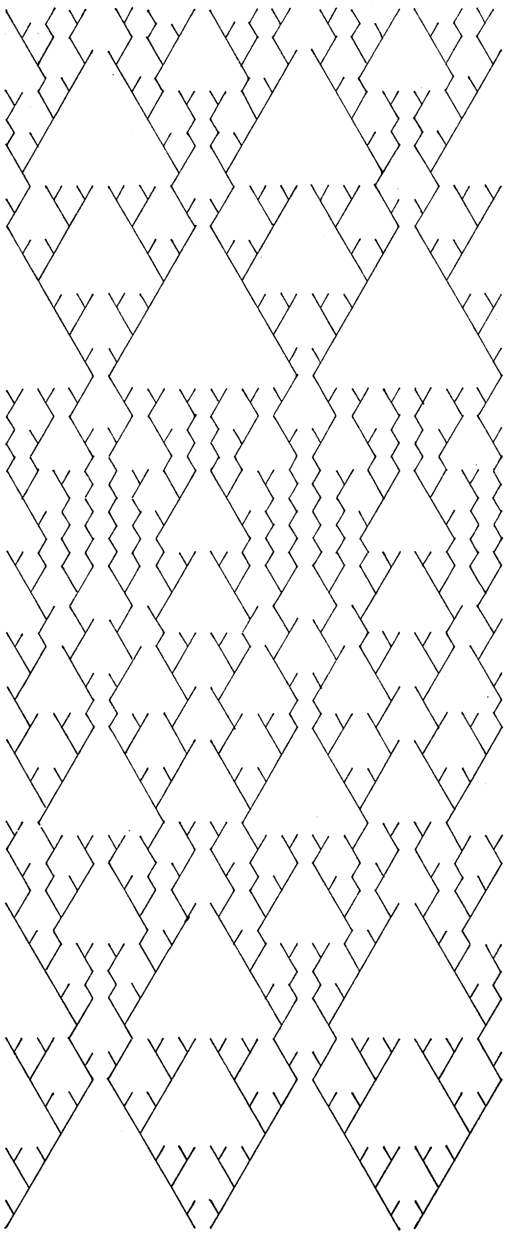

FIGURE 27. F18 13 × 63 R.

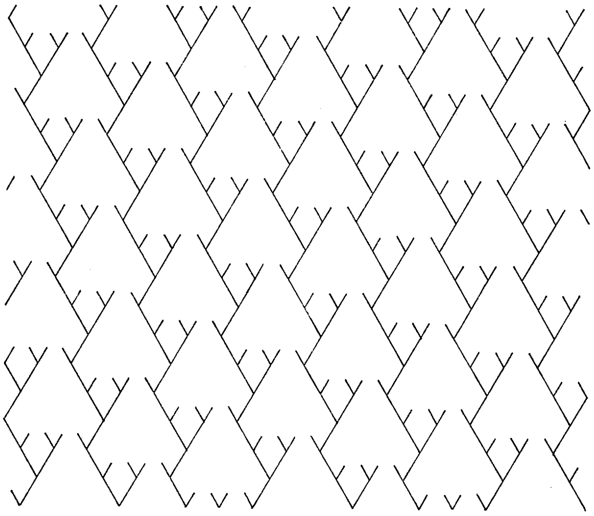

FIGURE 28. F21 14 × 2 S.

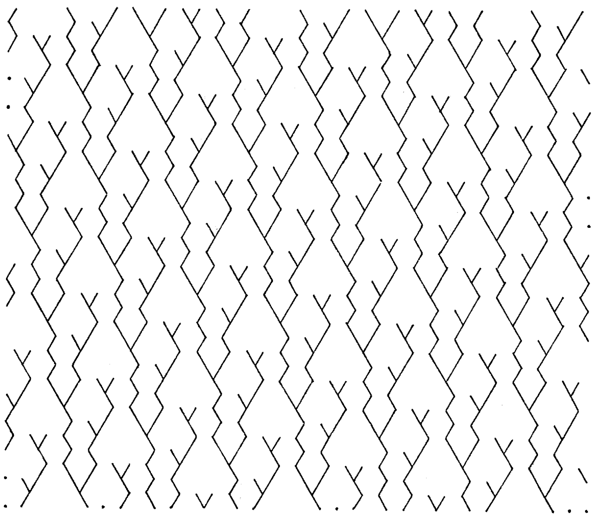

FIGURE 29. F22 14 × 2 S.

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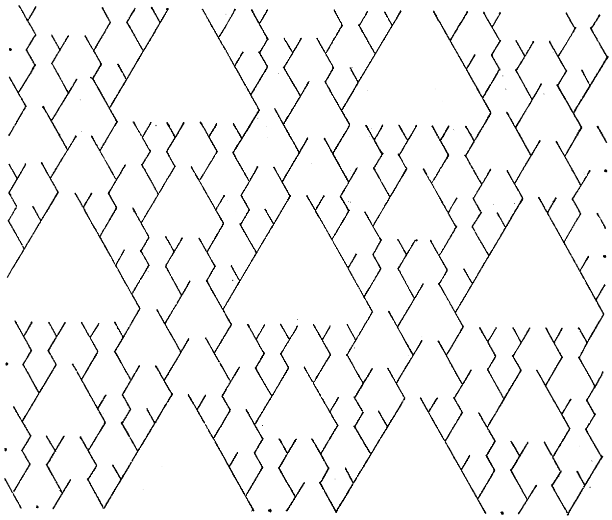

FIGURE 30. F23 14 × 14 S.

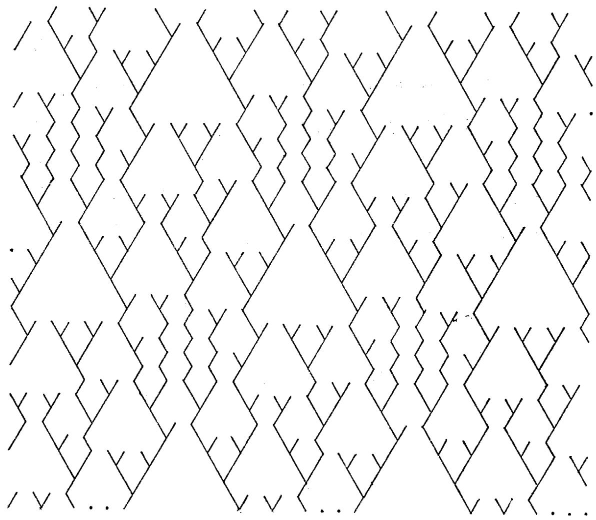

FIGURE 31. F24 14 × 14 S.

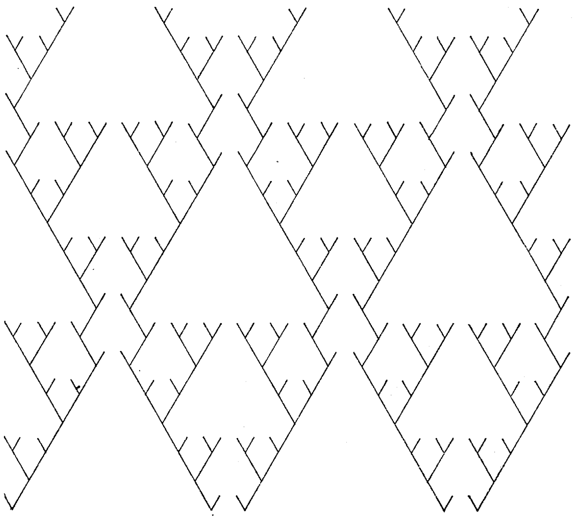

FIGURE 32. F25 14 × 14 T.

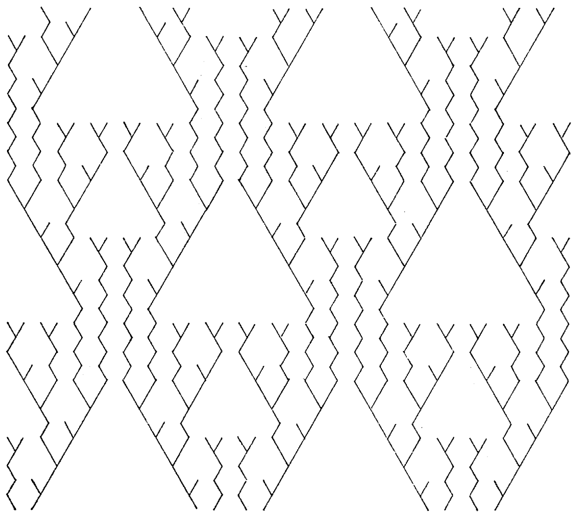

FIGURE 33. F26 14 × 14 T.

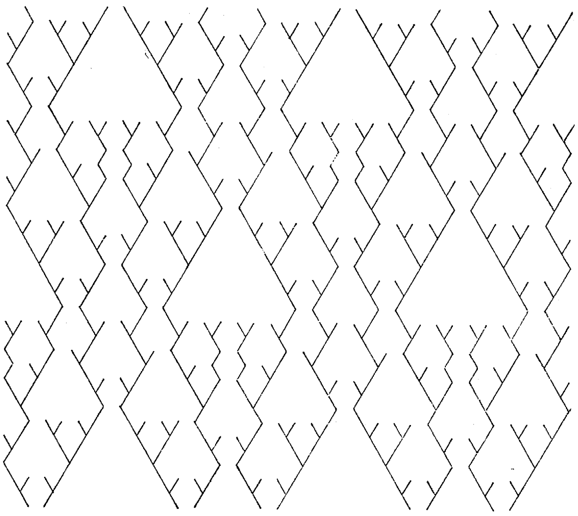

FIGURE 34. F27 14 × 14 R.

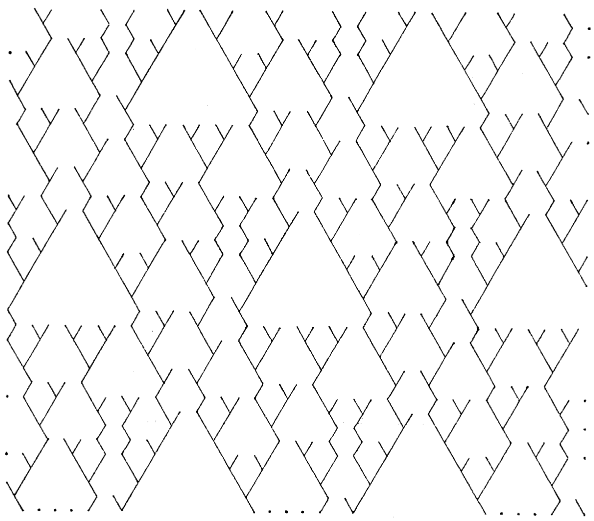

FIGURE 35. F28 14 × 14 R.

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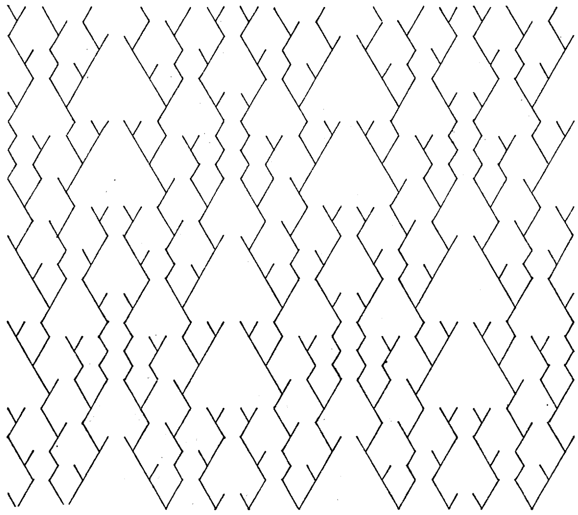

FIGURE 36. F29 14 × 14 R.

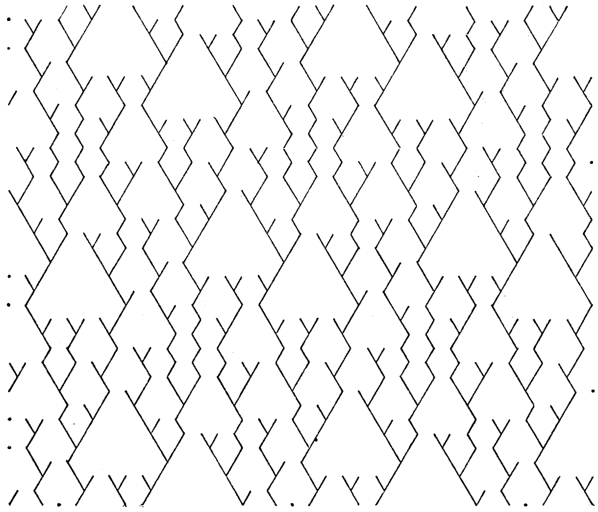

FIGURE 37. F30 14 × 14 R.

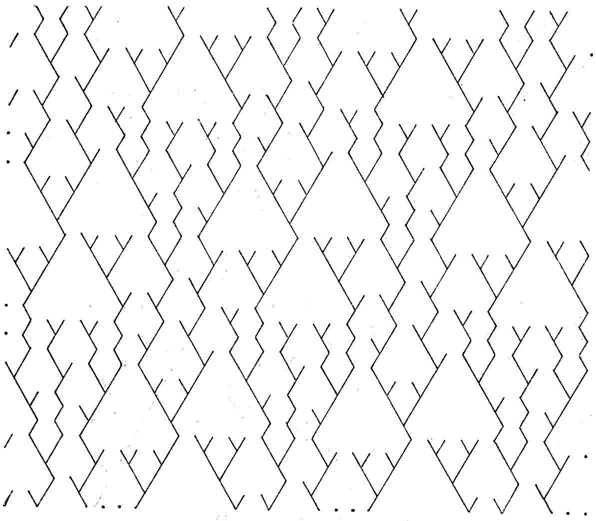

FIGURE 38. F31 14 × 14 S.

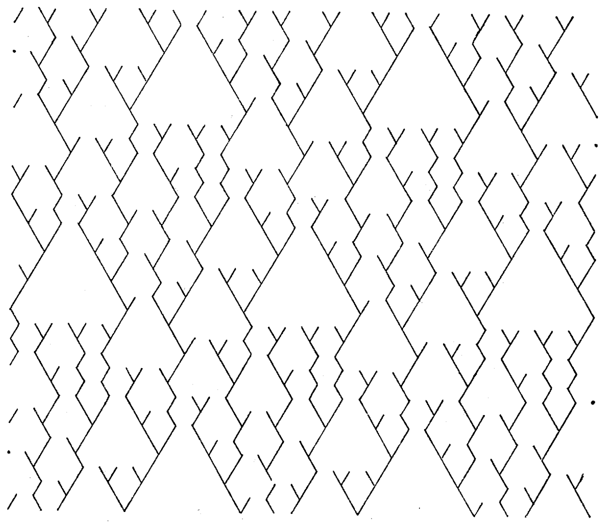

FIGURE 39. F32 14 × 14 U.

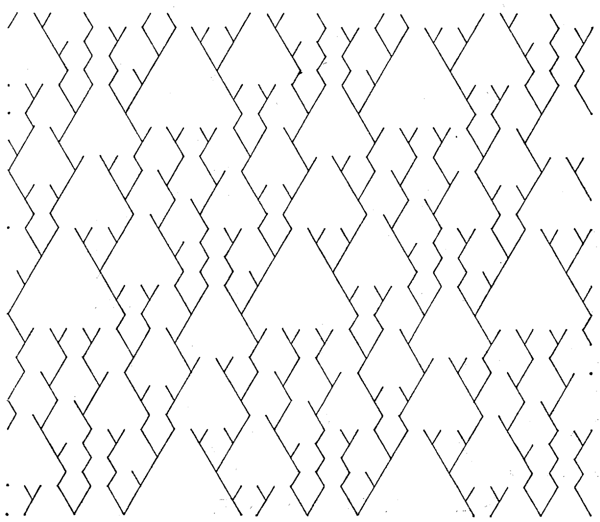

FIGURE 40. F33 14 × 14 U.

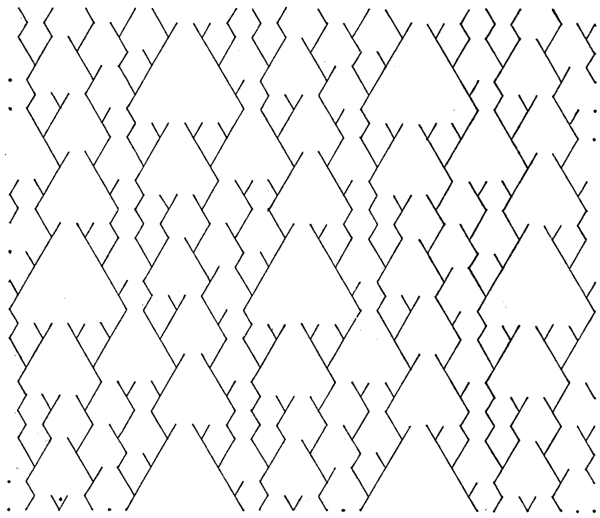

FIGURE 41. F34 14 × 14 U.

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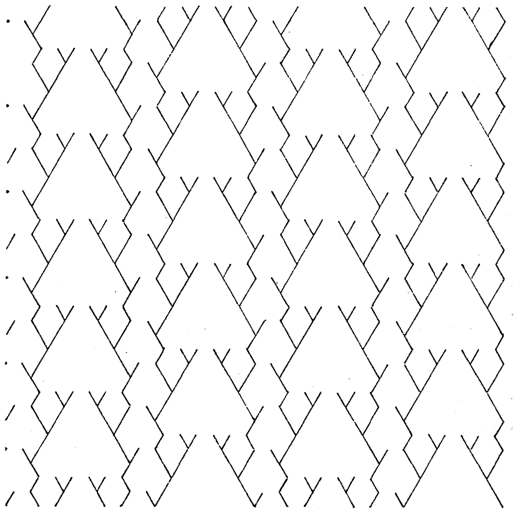

FIGURE 42. F36 15 × 3 R.

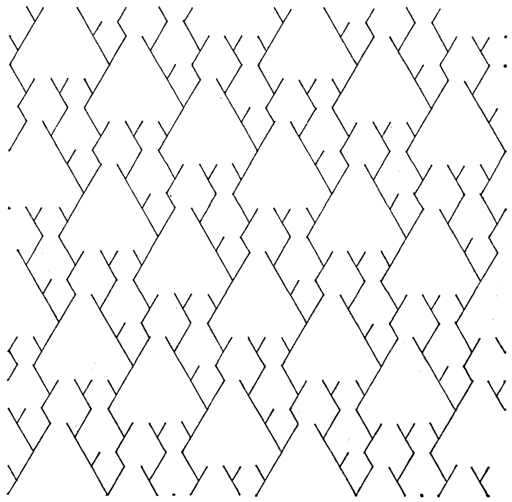

FIGURE 43. F37 15 × 3 R.

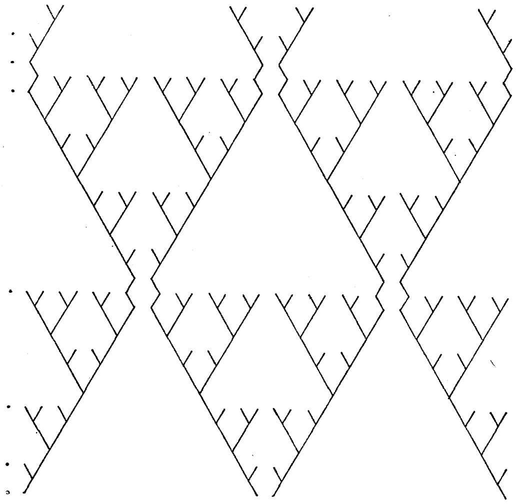

FIGURE 44. F54 15 × 15 T.

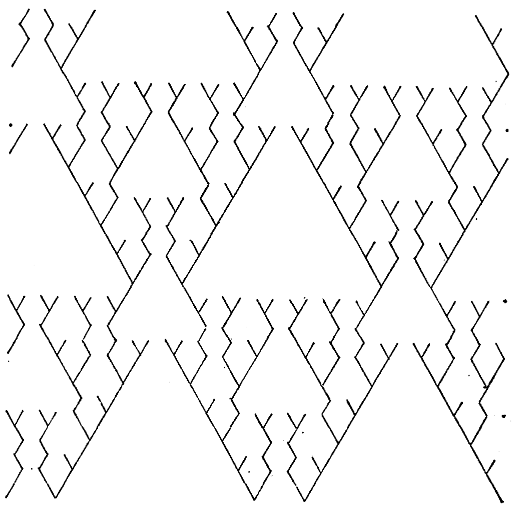

FIGURE 45. F45 15 × 15 T.

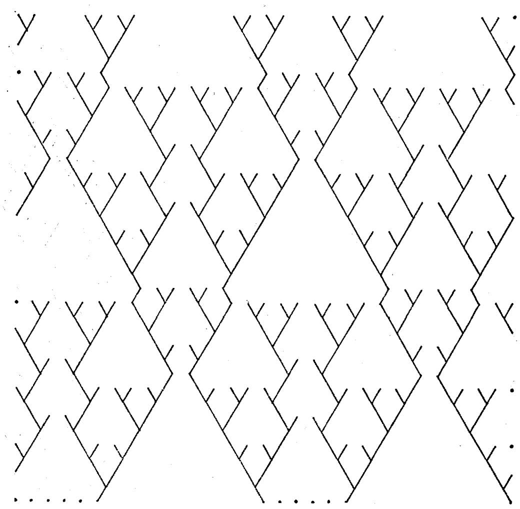

FIGURE 46. F46 15 × 15 T.

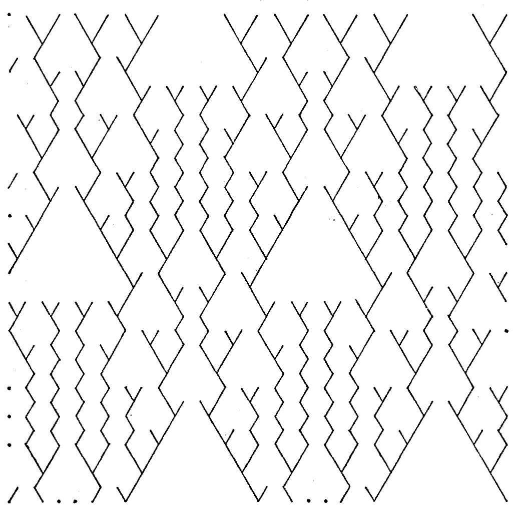

FIGURE 47. F47 15 × 15 T.

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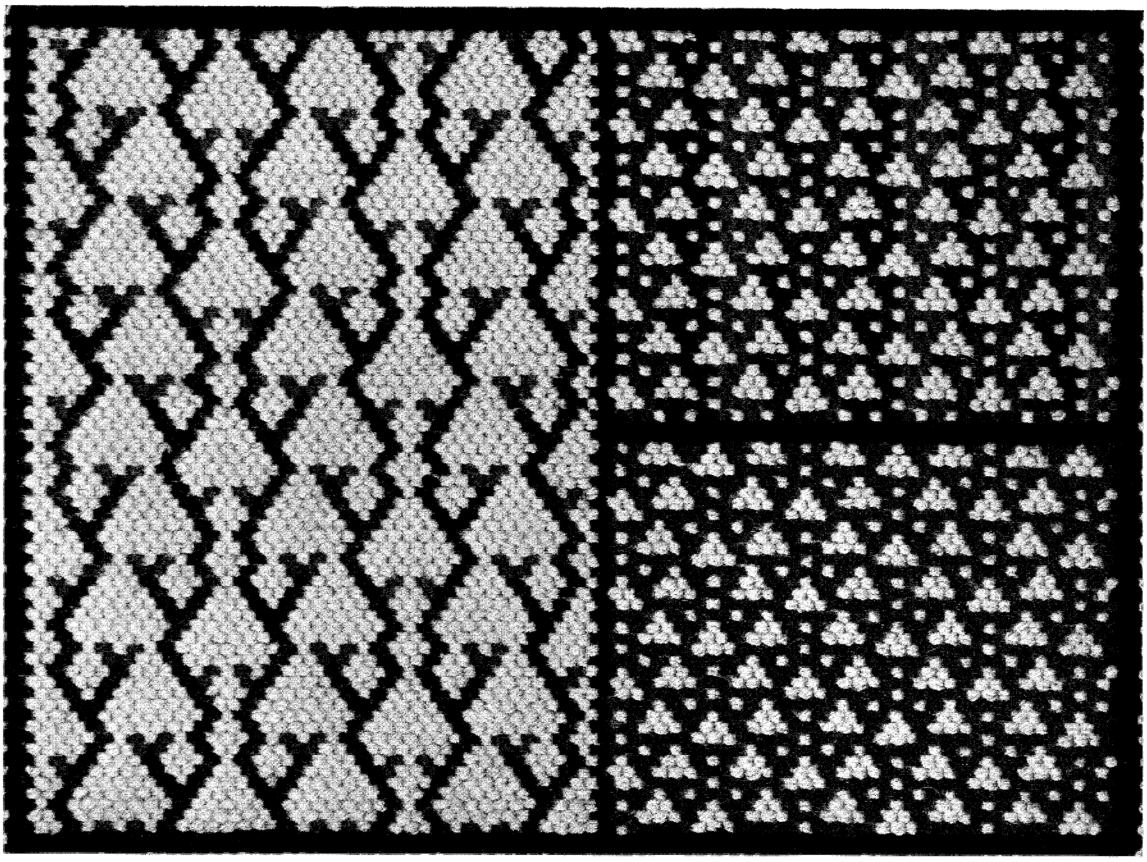

FIGURE 48. Three Tapestry designs based on F 17, figure 26.

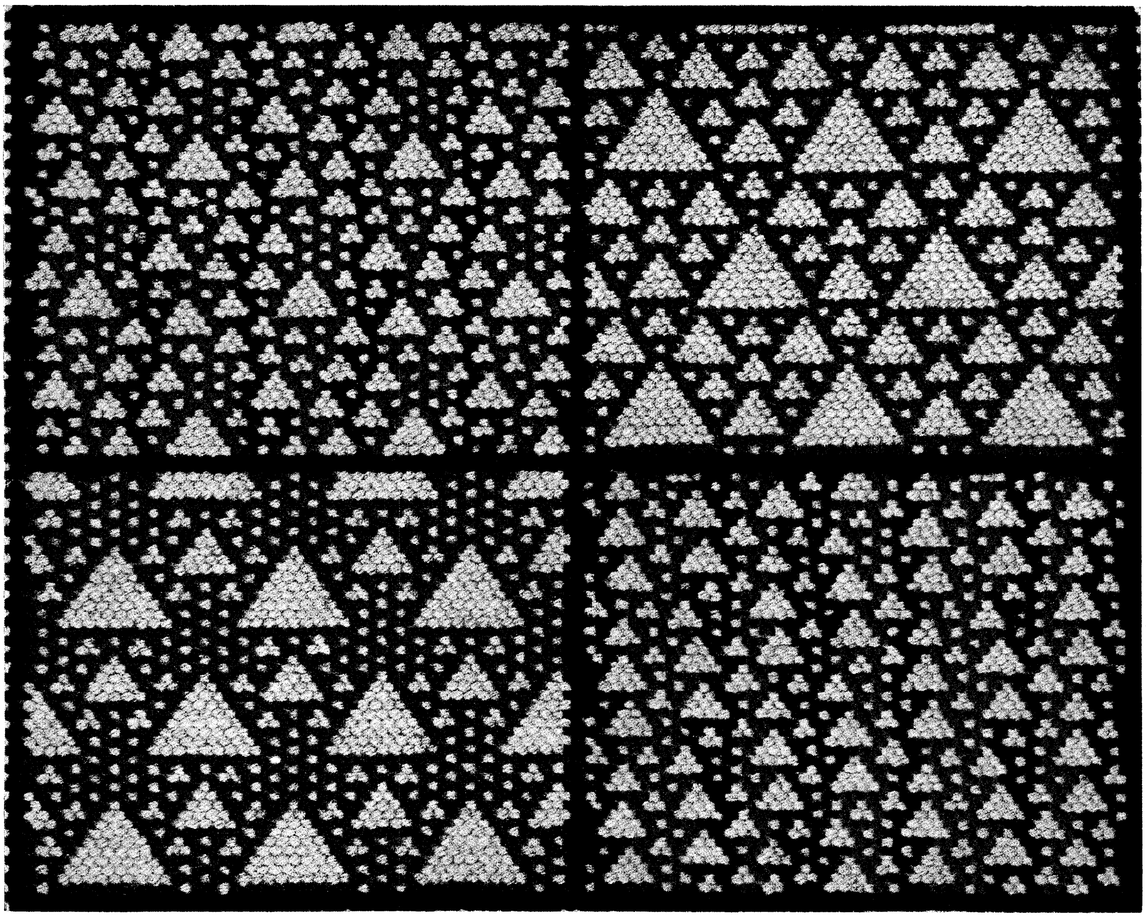

FIGURE 49. Four Tapestry designs based on F 26, F 46, figures 33 and 46 (both on red) and with $(n, m) = (17, 15)$ and $(24, 8)$.

(*Facing p.* 110)

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I am anxious to acknowledge the help of several students who made computer programs for me—S. R. Bourne, who produced most of the background designs on Titan, J. S. Elliott, who produced a PDP 7 display diagram (one result is pictured in Miller 1968, fig. 10), D. B. Webster, who produced all the material of table 4 on Titan, A. Henrici, who unearthed many of the symmetry properties and provided other ideas in discussion.

I am also indebted to Professor E. S. Selmer who stimulated my interest in periodic binary sequences in the first place, and thus helped me to reorganize a number of semi-independent investigations. He also made it possible for me to work with him in the University in Bergen for several months, where various problems were sorted out with him and his students. I must also thank Dr F. L. Bauer for a key idea leading to the matrix formulation. The Touch Tapestry tiles were kindly provided by Minimodels Ltd. Many others have also helped, and I wish to thank them also, even though not by name.

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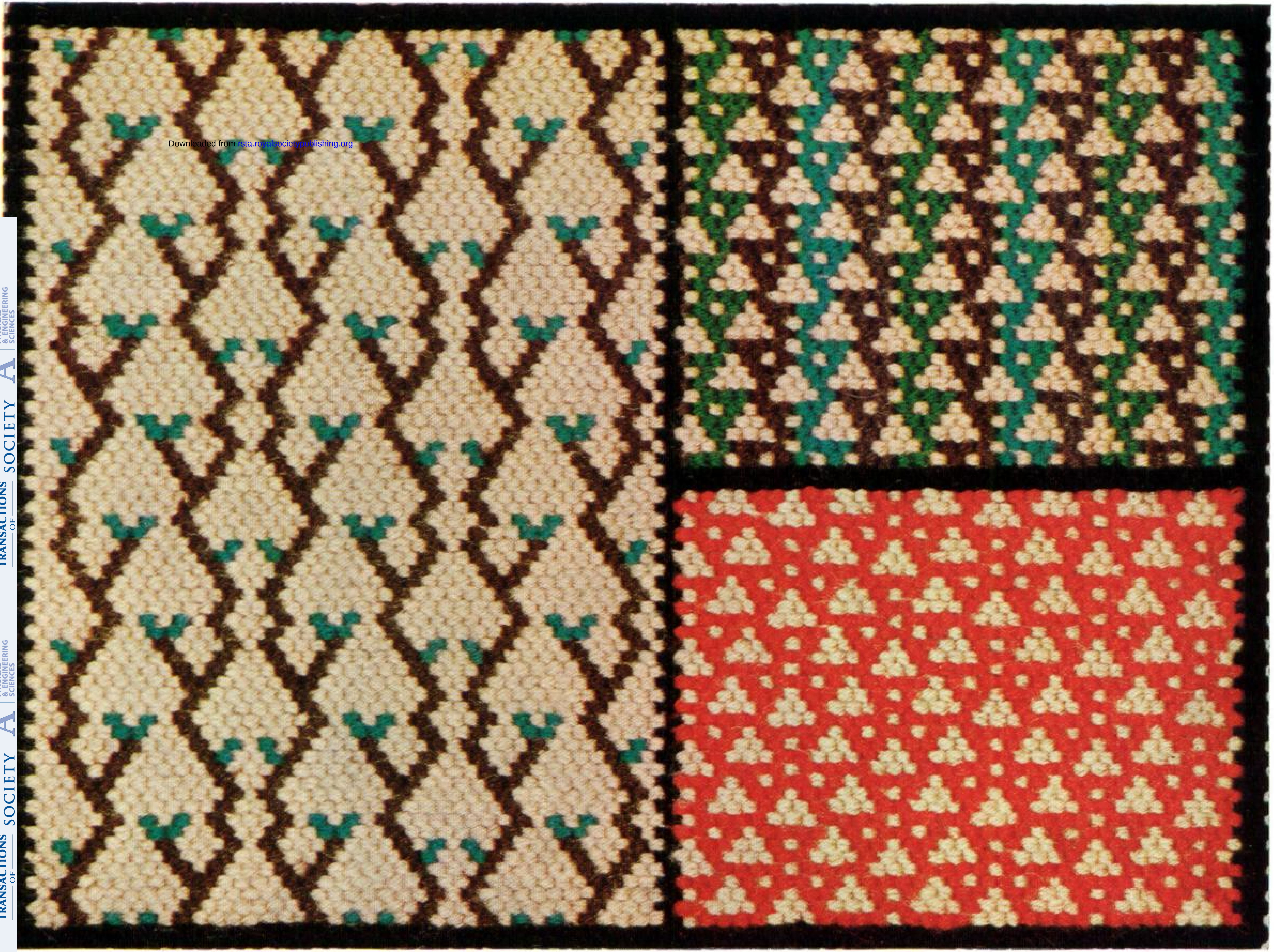

FIGURE 48. Three Tapestry designs based on F17, figure 26.

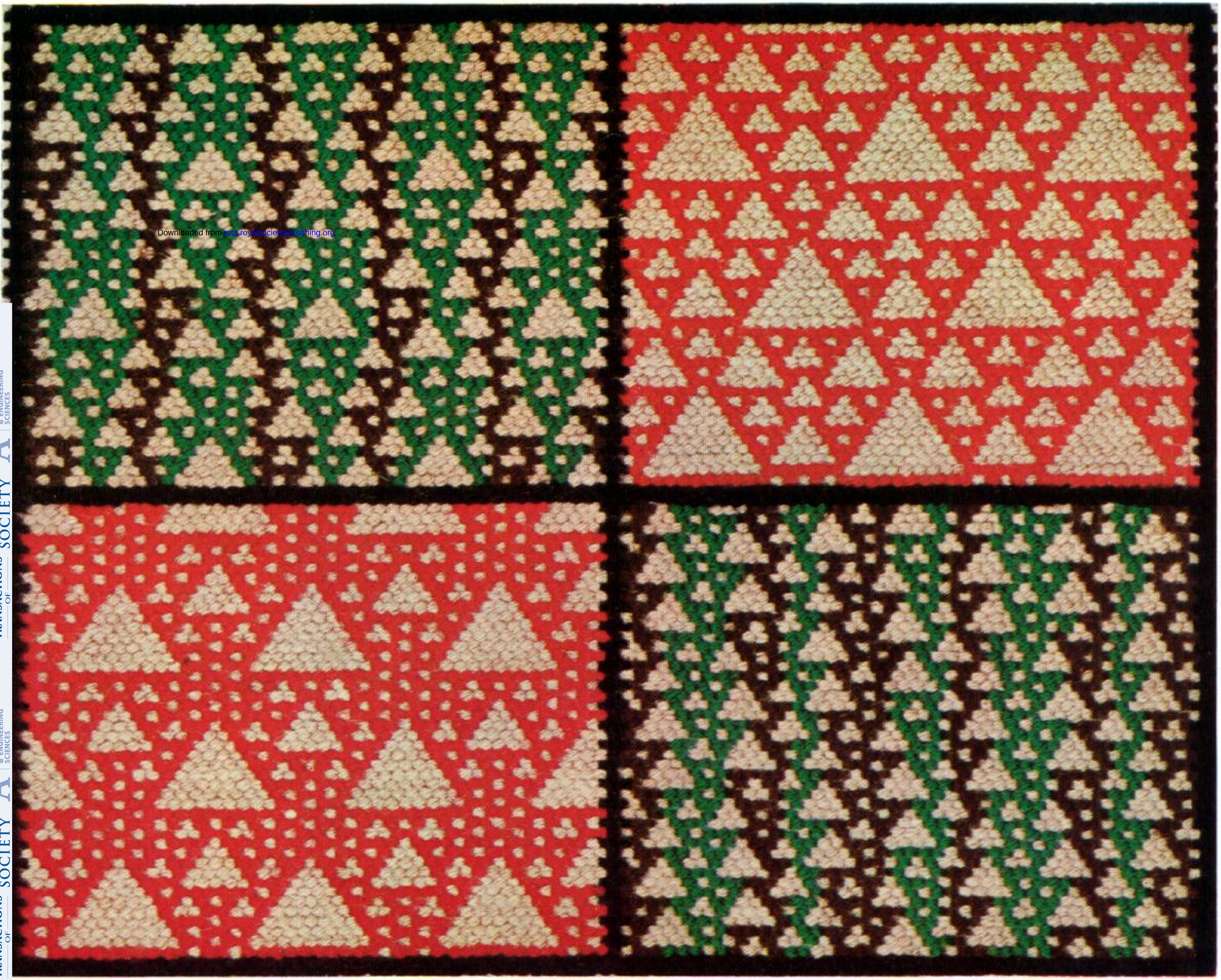

FIGURE 49. Four Tapestry designs based on F26, F46, figures 33 and 46 (both on red) and with $(n, m) = (17, 15)$ and $(24, 8)$.

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